Math 307M **Midterm 2A** **Winter 2018**

Your Name

Your Signature

Section (circle one) MA MB MC

Problem	Total Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

- This exam is closed book. No Notes. If you forget a formula, ask one of us.
- No graphing or symbolic calculators are allowed. You may not use cell phones during the exam.
- Show your work. Do not do computations in your head. Instead, write them out on the exam paper.
- **Place a box around YOUR FINAL ANSWER to each question.**
- If you need more room, use the backs of the pages and indicate that you have done so.
- If you are not sure what a question means, raise your hand and ask us.
- The hints are suggestions only.

1 (20 points)

Find the solution to

$$y'' + 6y' + 13y = 0$$
$$y(0) = 1 \qquad y'(0) = 1$$

and write it in amplitude-phase form: $y(t) = Ae^{rt}\cos(\omega t - \phi)$.

$$r^2 + 6r + 9 = -4$$
$$(r+3)^2 = -4$$
$$r + 3 = \pm 2i$$
$$r = -3 \pm 2i$$
$$y(t) = Ae^{-3t}\cos(2t) + Be^{-3t}\sin(2t)$$
$$1 = A$$
$$1 = -3A + 2B$$
$$4 = 2B \implies B = 2$$
$$y(t) = e^{-3t}\cos 2t + 2e^{-3t}\sin 2t$$
$$= e^{-3t}(1+2^2)^{1/2}\cos(2t - \operatorname{atan}(\tfrac{2}{1}))$$
$$\boxed{y(t) = e^{-3t}\sqrt{5}\cos(2t - \operatorname{atan}(2))}$$

$\boxed{2}$ (20 points)

Find the general solution to

$$y'' - 4y' + 4y = 2t^2 + 1 + e^t$$

$y_h'' - 4y_h' + 4y_h = 0$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$y_h = c_1 e^{2t} + c_2 t e^{2t}$

Seek y_p

$y_p'' - 4y_p' + 4y_p = 2t^2 + 1$

Seek $y_p = At^2 + Bt + C$

Homogeneous Question No

Seek y_p

$y_p'' - 4y_p' + 4y_p = e^t$

$y_p = Ae^t$

$y_p' = Ae^t$

$y_p'' = Ae^t$

Homogeneous Question No

$Ae^t - 4Ae^t + 4Ae^t = e^t$

$A = 1$

$y_p = e^t$

$4y_p = 4At^2 + 4Bt + 4C$

$-4y_p' = \cancel{2}^{-8}At + {}^{-4}B$

$+\ y_p'' = 2A$

$2t^2 + 1 = \underset{\overset{\shortparallel}{2}}{4A}t^2 + (\underset{\overset{\shortparallel}{0}}{4B - 8A})t + (\underset{\overset{\shortparallel}{1}}{4C - 4B + 2A})$

$\boxed{A = \frac{1}{2}}$

$4B - 8\frac{1}{2} = 0$

$\boxed{B = 1}$

$4C - 4 + 1 = 1$

$4C = 4$

$\boxed{C = 1}$

$$\boxed{y(t) = \frac{1}{2}t^2 + t + 1 + e^t + c_1 e^{2t} + c_2 t e^{2t}}$$

3 (20 points)

For linear differential equations of the form

$$t^2y'' + aty' + by = 0$$

we look for solutions of the form $y(t) = t^r$ instead of $y(t) = e^{rt}$. Find two solutions of the from $y(t) = t^r$ for the DE below and use them to solve the IVP

$$t^2y'' + ty' - 16y = 0$$
$$y(1) = 0 \qquad y'(1) = 1$$

$y = t^r$
$y' = rt^{r-1}$
$y'' = r(r-1)t^{r-2}$

$t^2 r(r-1)t^{r-2} + t\,rt^{r-1} - 16t^r = 0$

$r(r-1)t^r + rt^r - 16t^r = 0$

$r(r-1) + r - 16 = 0$

$r^2 - \cancel{r} + \cancel{r} - 16 = 0$

$r = \pm 4$

$y(t) = C_1 t^4 + C_2 t^{-4}$

$0 = y(1) = C_1 + C_2$

$1 = y'(1) = 4C_1 - 4C_2$

$0 = 4C_1 + 4C_2$
$+\ 1 = 4C_1 - 4C_2$

$1 = 8C_1$

$C_1 = \frac{1}{8}$

$C_2 = -\frac{1}{8}$

$$\boxed{y(t) = \frac{1}{8}t^4 - \frac{1}{8}t^{-4}}$$

4 (20 points) An undamped mass spring system is released from equilibrium with a velocity of 8 m/s. The mass is 1 kg and it oscillates with an amplitude of 2 meters. There is no forcing. Find the spring constant k.

$$\ddot{x} + kx = 0$$

$$x(0) = 0 \quad \dot{x}(0) = 8$$

$$x(t) = A\cos\sqrt{k}t + B\sin\sqrt{k}t$$

$$0 = A$$

$$8 = \sqrt{k}B$$

$$x(t) = \frac{8}{\sqrt{k}}\sin\sqrt{k}t$$

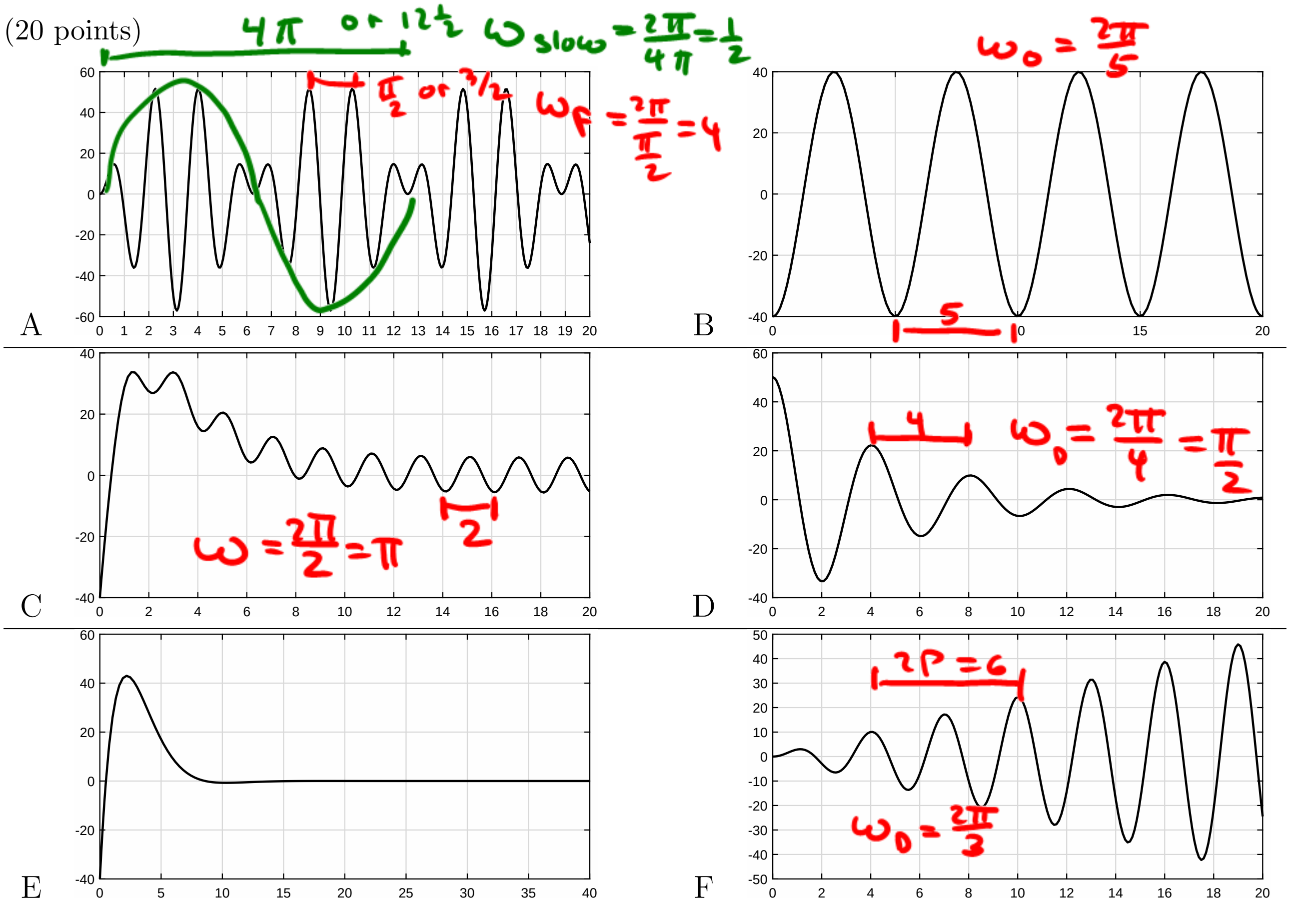
Amplitude

$$2 = \frac{8}{\sqrt{k}}$$

$$\sqrt{k} = 4$$

$$\boxed{k = 16}$$

$\boxed{5}$ (20 points)



Which graph shows a solution of a forced damped mass-spring system? What is the forcing frequency? C π

Which graph is a solution to an unforced overdamped mass-spring system?

E

Which graph is a solution to an unforced underdamped mass-spring system? What is the natural quasi-frequency? D $\frac{\pi}{2}$

Which graph is a solution to a system that is unforced and undamped? What is the natural frequency? B $\frac{2\pi}{5}$

Which graphs show solutions to forced and undamped mass-spring systems? What phenomena do they exhibit? *Extra credit: Write a plausible formula for each of these two solutions. I don't care if you write sine when it shold be cosine, but try to get the basic form and the frequencies approximately right.*

A Beats

$y(t) = 60\sin(4t)\sin(\tfrac{1}{2}t)$

F Resonance $y(t) = t\sin\frac{2\pi}{3}t$