1. (10 points) Find the general solution to the following second-order differential equation:

$$4y'' - 12y' + 9y = 9t.$$

To find the general solution to this nonhomogeneous differential equation, we recall that the solution to the nonhomogeneous DE is the sum of a particular solution to the full nonhomogenous DE and the general solution to the corresponding homogeneous DE.

First, the homogeneous part. The characteristic equation for the homogeneous part of the DE is $4r^2 - 12r + 9 = 0$; this has a double root at $r = \frac{3}{2}$. The general solution to the homogeneous differential equation is therefore

$$y = (c_1 + c_2 t)e^{\frac{3}{2}t}.$$

To find a particular solution to the nonhomogeneous DE, we use the method of undetermined coefficients. The forcing function for the DE is $9t$, which isn't a solution to the homogeneous part of th equation in any way, so we guess a general polynomial of the same degree, i.e. $Y = At + B$. Hence $Y' = A$ and $Y'' = 0$. Thus

$$\begin{aligned} 9t = 4Y'' - 12Y' + 9Y &= 4 \cdot 0 - 12(A) + 9(At + B) \\ &= 9At + (-12A + 9B). \end{aligned}$$

Equating coefficients gives us the system of equations $9A = 9$ and $-12A + 9B = 0$. The first tells us that $A = 1$, which in turn in the second equation implies that $-12 + 9B = 0$, i.e. $B = \frac{4}{3}$. W therefore have that

$$Y = t + \frac{4}{3}$$

Finally, the full general solution the differential equation is the sum of the general solution to the homogeneous DE and the particular solution to the nonhomogeneous DE. So we arrive at the solution

$$y = (c_1 + c_2 t)e^{\frac{3}{2}t} + t + \frac{4}{3}.$$

2. (10 points) Consider the differential equation

$$t^2y'' - 4ty' + 6y = 0.$$

One can check that $y_1(t) = t^2$ obeys this DE. Use the method of reduction of order or any other method of your choosing to find the solution subject to the initial conditions $y(1) = 1$, $y'(1) = 0$.

The method of reduction of order is not covered in all sections.

The method of reduction of order has that we guess that the general solution to the DE is equal to our know solution multiplied by an as-yet undetermined function $v(t)$; that is, let

$$y(t) = t^2v(t)$$

We now plug y back into the DE to solve for v and hence y. Specifically we have, using the product rule:

$$y' = 2tv + t^2v' \quad \text{and} \quad y'' = 2v + 4tv' + t^2v''.$$

Hence

$$\begin{aligned} t^2y'' - 4ty' + 6y &= t^2\left(2v + 4tv' + t^2v''\right) - 4t\left(2tv + t^2v'\right) + 6\left(t^2v\right) \\ &= 2t^2v + 4t^3v' + t^4v'' - 8t^2v - 4t^3v' + 6t^2v \\ &= t^4v'' \\ &= 0 \end{aligned}$$

We can divide through by t^4 at this point (it doesn't tell us anything useful) to obtain

$$v'' = 0.$$

The solution to this DE is any linear function in t, i.e.

$$v(t) = c_1t + c_2.$$

Hence

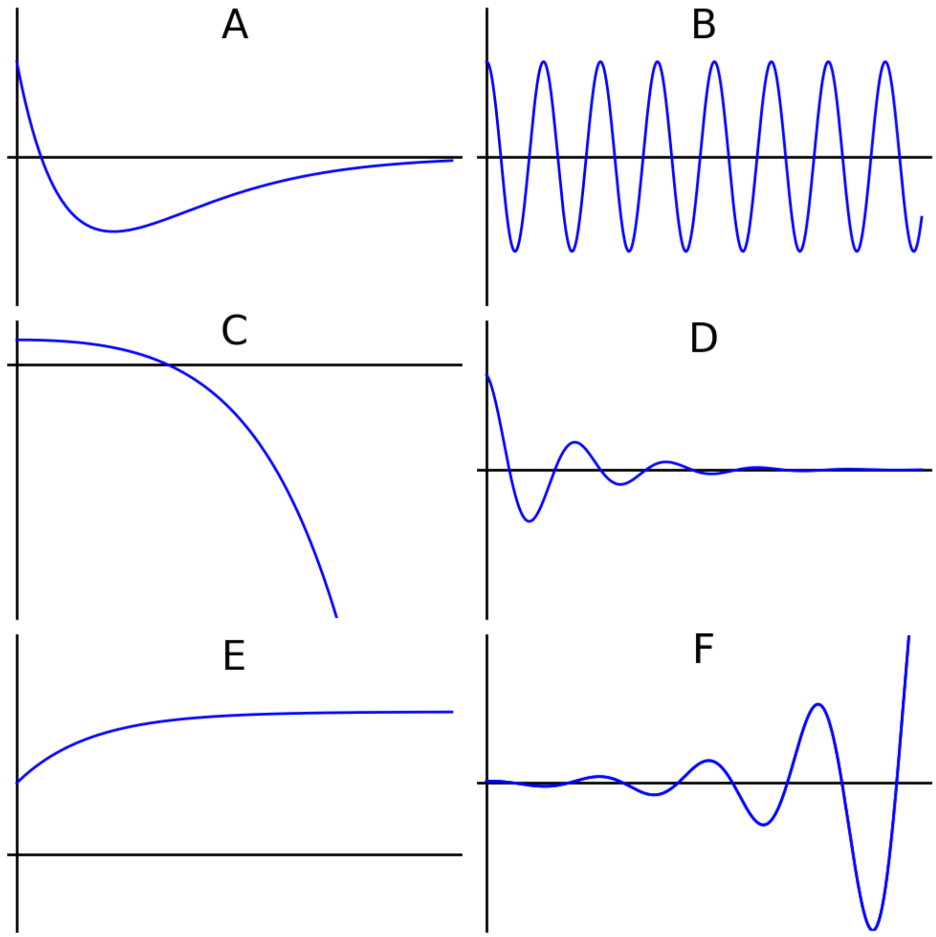
$$y(t) = t^2v(t) = c_1t^3 + c_2t^2$$

is the general solution to the original differential equation (we see that the known function $y_1(t) = t^2$ is included in the general solution, which is a good indication that our math is correct so far).

Now we apply initial conditions. We have $y(1) = 1$, so $c_1 + c_2 = 1$. The second IC $y'(1) = 0$ gives us $3c_1 + 2c_2 = 0$. Solving this system of equations yields $c_1 = -2$ and $c_2 = 3$. Hence we arrive at the solution to the initial value problem

$$y = -2t^3 + 3t^2 = (3 - 2t)t^2.$$

3. (10 points) Below are the graphs of six functions $y(t)$, with t and y being the horizontal and vertical axes respectively. The graphs are labeled A through F. The graphs are **not** all drawn to the same scale, and axis markings have been purposely omitted.



Each of the functions graphed above is a solution to exactly one of the six differential equations below. By analyzing the form of the equations' general solutions, write the letter of the graph next to the differential equation for which it is the solution. You do not need to show your work in this question to receive full credit.

1. $y'' - 3y' + 2y = 0$: C

   The characteristic equation for this DE is $r^2 - 3y + 2 = 0$, which has roots $r = 1$ and $r = 2$. Correspondingly the general solution to this differential equation is $y = c_1 e^t + c_2 e^{2t}$. Thus any nonzero solution must grow exponentially and not exhibit any oscillation. The only graph above that matches these criteria is graph number **C**.

2. $y'' + 16y = 0$: B

   The CE for this equation is $r^2 + 16 = 0$, which has roots $r = \pm 4i$. Correspondingly the general solution to this DE is $y = c_1 \cos(4t) + c_2 \sin(4t)$. Thus any nonzero solution must oscillate with constant amplitude. The only graph above exhibiting this behavior is graph number **B**.

3. $y'' - y' + \frac{3}{2}y = 0$: F

   The CE for this equation is $r^2 - r + \frac{3}{2} = 0$, which has roots $r = \frac{1}{2} \pm \frac{\sqrt{5}}{2}i$. Correspondingly the general solution to this DE is $y = e^{\frac{1}{2}t}(c_1\cos(\frac{\sqrt{5}}{2}t) + c_2\sin(\frac{\sqrt{5}}{2}t))$. Thus any nonzero solution must oscillate with exponentially growing amplitude. The only graph above exhibiting this behavior is graph number **F**.

4. $y'' + y' + \frac{3}{2}y = 0$: D

   The CE for this equation is $r^2 + r + \frac{3}{2} = 0$, which has roots $r = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}i$. Correspondingly the general solution to this DE is $y = e^{-\frac{1}{2}t}(c_1\cos(\frac{\sqrt{5}}{2}t) + c_2\sin(\frac{\sqrt{5}}{2}t))$. Thus any nonzero solution must oscillate with exponentially decaying amplitude. The only graph above exhibiting this behavior is graph number **D**.

5. $y'' + y' + \frac{1}{4}y = 0$: A

   The CE for this equation is $r^2 + r + \frac{1}{4} = 0$, which has a double root at $r = -\frac{1}{2}$. Correspondingly the general solution to this DE is $y = (c_1t + c_2)e^{-\frac{1}{2}t}$. Thus a nonzero solution to this differential equation may initially grow or it may cross the equilibrium point, but it must eventually decay to zero without exhibiting oscillation. The only graph above with this behavior is graph number **A**.

6. $y'' + 2y' = 0$: E

   The CE for this equation is $r^2 + 2r = 0$, which has a roots $r = 0$ and $r = -2$. Correspondingly the general solution to this DE is $y = c_1 + c_2e^{-2t}$. Thus a nonzero solution to this differential equation must asymptote to a possibly nonzero constant without oscillating. The only graph above with this behavior is graph number **E**.

4. (10 total points) A series circuit contains a capacitor of 2×10^{-4} F, an inductor of 2 H, and a resistor of R ohms. Consider the differential equation governing the charge $Q(t)$ on the capacitor as a function of time, where Q is in Coulombs and t in seconds.

   (a) (2 points) We ascertain that R is such that the system exhibits critical damping. Find R.

   We use the series circuit differential equation that we developed in class, i.e.

$$LQ'' + RQ' + \frac{1}{C}Q = E(t),$$

   where $Q(t)$ is the charge on the capacitor as a function of time, and for us $L = 2$, $1/C = 1/(2\times 10^{-4}) = 5000$ and $E(t)$ is as yet unspecified. Hence we have the differential equation

$$2Q'' + RQ' + 5000Q = E(t).$$

   The corresponding characteristic equation is $2r^2 + Rr + 5000 = 0$. Now critical damping occurs when the discriminant ('$b^2 - 4ac$') is zero. Thus we must have $R^2 - 4\cdot 2\cdot 5000 = 0$, or $R^2 = 40000$. Taking the positive square root (we know resistance is positive) yields

$$R = 200\ \ \Omega.$$

   (b) (6 points) The initial charge on the capacitor is zero and there is no initial current. Starting at time $t = 0$, a constant external voltage of 100 volts is applied, where t is in seconds. Find the the charge on the capacitor as a function of time. What is the amplitude of the steady-state response?

   Using the above information and part (a), we arrive at the the initial value problem

$$2Q'' + 200Q' + 5000Q = 100, \qquad Q(0) = 0, Q'(0) = 0.$$

   The characteristic equation is $2r^2 + 200r + 5000 = 0$ or $2(r+50)^2 = 0$, to the CE has a repeated root of $r = 50$. Hence the general solution to the homogeneous DE is

$$y = (c_1 + c_2 t)e^{-50t}.$$

   Since the forcing function is constant, we guess a constant for the particular solution, i.e. $Y = A$ for some value of A. The $Y' = Y'' = 0$, so plugging this back into the DE gives us $5000A = 100$. Hence $A = \frac{1}{50}$. The full general solution to the nonhomogeneous DE is

$$Q = (c_1 + c_2 t)e^{-50t} + \frac{1}{50}.$$

   Now we apply initial conditions. $Q(0) = 0$ implies that $c_1 = -\frac{1}{50}$. We have $Q' = -50c_1e^{-50t} + c_2e^{-50t} - 50c_2te^{-50t}$, so $Q'(0)$ implies that $-50c_1 + c_2 = 0$. Thus $c_2 = -1$. Hence the solution to the initial value problem is

$$Q(t) = \frac{1}{50} - \frac{1}{50}e^{-50t} - te^{-50t}.$$

   We see from this that the steady-state response is constant at $Q = \frac{1}{50}$ Coulombs, i.e. the amplitude of the forced response is $\frac{1}{50}$ Coulombs.

(c) (2 points) The external voltage is now changed to $100\cos(50t)$ volts. Will the amplitude of the steady-state response increase, decrease or stay the same compared to what you found in part (b)? Justify your answer.

Even though $\omega = 50$ rad/sec is the natural frequency of the system, the amplitude of the steady-state response will decrease. This is because we are in the state of critical critical damping; $\Gamma = \frac{\gamma^2}{km}$ (or correspondingly $\frac{R^2C}{L}$ in the case of the series circuit) is equal to 4, i.e. damping is very large compared to the other coefficients in the equation. We know that resonance (an increase in the forced response amplitude) only occurs for $\Gamma < 2$, so we will definitely see a decrease in the amplitude of the steady-state solution.

Alternatively one can use the formula for the steady state amplitude. For the DE $my'' + \gamma y' + ky = F_0\cos(\omega t)$ we have forced response amplitude R as

$$R = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \gamma^2\omega^2}}.$$

Plugging in $F_0 = 100$, $m = 2$, $\gamma = 200$, $k = 5000$ and $\omega = 50$ gives us

$$\begin{aligned} R &= \frac{100}{\sqrt{(5000 - 2\cdot 50^2)^2 + 200^2\cdot 50^2}} \\ &= \frac{100}{\sqrt{0 + 10000^2}} \\ &= \frac{1}{100} \text{ Coulombs.} \end{aligned}$$

This is quite clearly less than the previous forced response amplitude of $\frac{1}{50}$ Coulombs.

5. (10 total points) A $\frac{1}{4}$ kg mass is placed on a flat frictionless surface and attached to a horizontal spring. It takes 4 N of force to move the mass 36 cm to the right of its equilibrium position. The mass starts at rest in its equilibrium position. Starting at time $t = 0$ seconds a horizontal force of $0.41\cos(7t)$ Newtons acts on the mass. Friction in this problem is negligible.

   (a) (3 points) Formulate an initial value problem that describes the position of the mass at time t.

   Let y be the position of the mass, where positive y points to the right. The differential equation is then in the form
$$my'' + \gamma y' + ky = F_0 \cos(\omega t).$$
   For us $m = \frac{1}{4}$, $\gamma = 0$, $F_0 = \frac{41}{100}$ and $\omega = 7$, so it remains to find the spring constant k. Now a displacement of 0.36 meters to the right results in a restoring force of $F_s = -4$ N; since $F_s = -ky$ we get $-4 = -k \cdot \frac{36}{100}$, so $k = \frac{100}{9}$.

   Finally the mass is initially at rest in its equilibrium condition, so we have the initial conditions $y(0) = 0$ and $y'(0) = 0$. Therefore we arrive at the IVP
$$\frac{1}{4}y'' + \frac{100}{9}y = \frac{41}{100}\cos(7t), \qquad y(0) = 0,\ y'(0) = 0.$$

   (b) (5 points) Solve the above initial value problem to find the position of the mass at time t. You may use known formulae to save time, but be sure to indicate if you are quoting a formula you've seen in class.

   One could solve this initial value problem in the usual way using the method of undetermined coefficients. However, consulting our notes we see that this is precisely the case where we get beats, i.e. we can write the solution to this IVP
$$y = [R\sin(\omega_1 t)]\sin(\omega_2 t),$$
   where the first sinusoidal term oscillates much more slowly than the second sinusoidal term. Specifically, since we worked it out on the board in class in full generality it's completely okay to just quote the formula for the solution:
$$y = \left[\frac{2F_0}{m(\omega_0^2 - \omega^2)}\sin\left(\frac{1}{2}(\omega_0 - \omega)t\right)\right]\sin\left(\frac{1}{2}(\omega_0 + \omega)t\right),$$
   where $\omega_0 = \sqrt{\frac{k}{m}}$ is the system's natural frequency. For us $\omega_0 = \sqrt{\frac{100/9}{1/4}} = \frac{20}{3}$. Hence
$$\begin{aligned} y &= \left[\frac{2 \cdot \frac{41}{100}}{\frac{1}{4}\left(\left(\frac{20}{3}\right)^2 - 7^2\right)}\sin\left(\frac{1}{2}\left(\frac{20}{3} - 7\right)t\right)\right]\sin\left(\frac{1}{2}\left(\frac{20}{3} + 7\right)t\right) \\ &= \left[-\frac{18}{25}\sin\left(-\frac{1}{6}t\right)\right]\sin\left(\frac{41}{6}t\right) \end{aligned}$$
   So after canceling minus signs we get the solution
$$y = \frac{18}{25}\sin\left(\frac{1}{6}t\right)\sin\left(\frac{41}{6}t\right).$$
   If you prefer decimals, the solution can be written as
$$y = 0.72\sin(0.1833t)\sin(6.833t).$$

(c) (2 points) What is the maximum distance the mass achieves from its equilibrium position?

This is easy once we put the solution in the form $y = R\cos(\omega t - \delta)$, as the maximum displacement is just the constant R. But this is exactly what we did in the previous part of the question, so we conclude that the maximum distance from its equilibrium position achieved by the mass is

$$R = \frac{18}{25} = 0.72 \text{ meters.}$$

(d) (Bonus: 3 points) Estimate the maximum amount of kinetic energy that the mass will have.

Kinetic energy $E_k = \frac{1}{2}mv^2$, so kinetic energy is maximized when $v = y'$ is maximum in magnitude. Since we have a formula for y for all t is quite possible to compute y' and solve for where it is a maximum or a minimum; however this is tedious and will take a long time by hand. The simplification we can make is to realize that we can think of y as a sinusoidal function oscillating at frequency $\omega = \frac{41}{6}$ rad/sec, but whose amplitude is varying slowly according to $\sin(\frac{1}{6}t)$.

Since the derivative of a sinusoidal function scales with the amplitude thereof, we expect maximum amplitude velocity to occur at or near the point where the amplitude of oscillation is a maximum, i.e. where $\sin(\frac{1}{6}t) = 1$. Round about that point in time we have

$$y \approx \frac{18}{25}\sin\left(\frac{41}{6}t\right),$$

so $y' \approx \frac{18}{25} \cdot \frac{41}{6}\cos\left(\frac{41}{6}t\right) = \frac{123}{25}\cos\left(\frac{41}{6}t\right)$. The maximum magnitude that this function achieves is clearly $\frac{123}{25}$, since the cosine part oscillates between -1 and 1. Hence $v_{max} \approx \frac{123}{25} = 4.92$ m/s.

It follows that

$$E_{max} = \frac{1}{2}mv_{max}^2 \approx \frac{1}{2} \cdot \frac{1}{4}\left(\frac{123}{25}\right)^2 = \frac{15129}{5000} = 3.0258 \text{ Joules.}$$

This estimate is very close to the true maximum energy (found numerically) of 3.0214 Joules.