1. (10 pts)

(a) Find the explicit solution to $\frac{dy}{dt} = 5t^4(y-3)^2$ with $y(0) = 4$.

$$\int \frac{1}{(y-3)^2}\,dy = \int 5t^4\,dt$$

NOT EQUILIBRIUM: $y(t) = 3$

$$\frac{-1}{y-3} = t^5 + C$$

$$y - 3 = \frac{-1}{t^5+C}$$

$$y = 3 - \frac{1}{t^5+C}$$

$$\boxed{y(t) = 3 - \frac{1}{t^5-1}}$$

$$y(0) = 4 \Rightarrow \quad 4 = 3 - \frac{1}{C} \Rightarrow C = -1$$

(b) Find the explicit solution to $\frac{dy}{dx} = x - \frac{3y}{x}$ with $y(1) = \frac{21}{5}$. (You may assume $x > 0$).

$$\frac{dy}{dx} + \frac{3}{x}y = x$$

$$e^{\int \frac{3}{x}dx} \overset{C=0}{=} e^{3\ln|x|} = x^3$$

$$x^3\frac{dy}{dx} + 3x^2y = x^4$$

$$\frac{d}{dx}(x^3y) = x^4$$

$$x^3y = \int x^4\,dx$$

$$x^3y = \frac{1}{5}x^5 + C$$

$$y = \frac{1}{5}x^2 + \frac{C}{x^3}$$

$$\boxed{y(x) = \frac{1}{5}x^2 + \frac{4}{x^3}}$$

$$y(1) = \frac{21}{5} \Rightarrow \quad \frac{21}{5} = \frac{1}{5} + C \quad C = 4$$

We don't teach this method anymore. Ignore part a.

2. (10 pts)

(a) Find a solution to $\frac{dy}{dx} = \frac{2x+y}{3+3y^2-x}$ with $y(0) = 1$. (Leave your answer in implicit form)

EXACT!

$$(-2x-y) + (3+3y^2-x)\frac{dy}{dx} = 0$$

$$\frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x} \quad ✓$$

$$\int -2x-y\,dx = -x^2 - xy + C_1(y)$$

$$\int 3+3y^2-x\,dy = 3y + y^3 - xy + C_2(x)$$

$$3y + y^3 - x^2 - xy = C$$

$$y(0)=1 \Rightarrow \quad 3+1-0-0 = C \Rightarrow C = 4$$

$$\boxed{3y + y^3 - x^2 - xy = 4}$$

(b) Find two different solutions to $\frac{dy}{dt} = 4t\sqrt{y-5}$ with $y(0) = 5$.

JUST LIKE A TEST PREP PROBLEM!

ONE SOL'N: $\boxed{y(t) = 5}$

ANOTHER: $\int (y-5)^{-1/2}\,dy = \int 4t\,dt$

$$2(y-5)^{1/2} = 2t^2 + C$$

$$(y-5)^{1/2} = t^2 + \tfrac{1}{2}C$$

$$y - 5 = (t^2 + \tfrac{1}{2}C)^2 \qquad D = \tfrac{1}{2}C$$

$$y = 5 + (t^2 + D)^2$$

$$y(0) = 5 \Rightarrow \quad D = 0$$

$$\boxed{y(t) = 5 + t^4}$$

EXACT SAME REASON AS A HOMEWORK PROBLEM!

$$\frac{\partial}{\partial y}\left(4t\sqrt{y-5}\right) = \frac{4t}{\sqrt{y-5}}$$

IS NOT CONTINOUS WHEN $y = 5$ AND THE INITIAL CONDITION HAS $y(0) = 5$!

3. (10 pts)

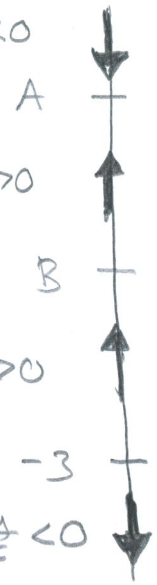
(a) For this question, assume A and B are constants with $A > B > 0$. Find all equilibrium solutions for $\frac{dy}{dt} = (A - y)(y - B)^2(y + 3)$ and classify each as stable, unstable or semistable.

$y > A \Rightarrow \frac{dy}{dt} < 0$

$B < y < A \Rightarrow \frac{dy}{dt} > 0$

$-3 < y < B \Rightarrow \frac{dy}{dt} > 0$

$y < -3 \Rightarrow \frac{dy}{dt} < 0$



$y(t) = A$ IS STABLE

$y(t) = B$ IS SEMISTABLE

$y(t) = -3$ IS UNSTABLE

(b) A local pond has an initial volume of 5,000 liters. Water enters the pond through a stream at 400 liters/day and water leaves the pond through a different stream at 600 liters/day. Salt, from the salting of roads, gets into the incoming stream giving it a salt concentration of 0.03 kg/liter. Initially, the pond contains no salt. Let $y(t)$ is the amount of salt in the pond at time t days.
Give the differential equation and initial condition for $y(t)$.
(DO NOT SOLVE, JUST SET UP)

400 − 600

VOLUME OF THE POND $= V(t) = 5000 - 200t$

$$\frac{dy}{dt} = \underset{\text{kg}/\text{L}}{(0.03)}\underset{\text{L}/\text{day}}{(400)} - \underset{\text{kg}/\text{L}}{\left(\frac{y}{5000-200t}\right)}\underset{\text{L}/\text{day}}{(600)}$$

$$\boxed{\frac{dy}{dt} = 12 - \frac{3y}{25-t}, \quad y(0) = 0}$$

YOU SAW THIS APPLICATION IN THE 2.3 AND 2.5 HOMEWORK.
THIS IS ALSO NEARLY IDENTICAL TO A 2.7 PROBLEM I ASSIGNED.

4. (10 pts) A large cylindrical water tank is 10 meters tall and has a small hole in the bottom that is leaking out water. Water is also being pumped in at a constant rate. You are given that

$$\frac{dh}{dt} = 1.5 - 0.5\sqrt{h},$$

where $h = h(t)$ is the height, in meters, of the water in the tank at time t minutes. The tank starts at a height of 4 meters (so $h(0) = 4$ meters).

(a) Approximate the height of the water in the tank in 1 minute using Euler's method with step size 0.5 minutes. In other words, approximate $h(1)$.
(Keep all your work accurate to four digits after the decimal).

$t_0 = 0, \; h_0 = 4$

$$\left.\begin{aligned} t_1 = 0.5, \; h_1 &= 4 + (1.5 - 0.5\sqrt{4})\,0.5 \\ &= 4 + 0.5 \cdot 0.5 = 4.25 \end{aligned}\right\} \Rightarrow h(0.5) \approx 4.25$$

$$\begin{aligned} t_2 = 1, \; h_2 &= 4.25 + (1.5 - 0.5\sqrt{4.25})\,0.5 \\ &\approx 4.25 + 0.469224 \cdot 0.5 \\ &\approx 4.48461 \end{aligned}$$

$$\boxed{h(1) \approx 4.4846 \text{ meters}}$$

(b) Does the tank overflow, completely drain out, or approach a limiting height (if so what is that height)? In other words, find $\lim_{t\to\infty} h(t)$ and explain your work.

THINK EQUILIBRIUM ANALYSIS!

EQUILIBRIUM SOL'NS ? : $1.5 - 0.5\sqrt{h} = 0$

$\Rightarrow 1.5 = 0.5\sqrt{h}$

$\Rightarrow 3 = \sqrt{h}$

$\Rightarrow h = 9$

$\longrightarrow h(t) = 9$

IN ADDITION, if $h < 9$, then $\frac{dh}{dt} = 1.5 - 0.5\sqrt{h}$ is positive.

THUS, STARTING AT $h(0) = 4$ THE HEIGHT WILL INCREASE AND APPROACH $\boxed{9 \text{ meters}}$

NOTE: $h(t) = 9$ IS STABLE

WE DISCUSSED THIS APPLICATION ON THE FIRST DAY OF CLASS AND DURING THE 2.3 LECTURE. ALSO SEE THE "FIRST ORDER APPLICATIONS" AND "2.3 REVIEW". WAS ALSO IN HOMEWORK.

5. (10 pts) A person wants to buy a home and they are going to get a home loan. The person can afford to make payments of \$20,000 per year. Assume payments are spread out and paid continuously throughout the year. Also, assume that current interest rates are 5%, compounded continuously.

Just like we did in class and in homework, assume that the rate due to interest is proportional to the balance (with proportionality $r = 0.05$) and that the payments are removed from the balance at a constant rate.

Let $y(t)$ be the loan balance after t years.

(a) Set up the differential equation AND solve to get $y(t)$.
Your answer will involve a constant of integration.

$$\boxed{\frac{dy}{dt} = ry - k = \underbrace{0.05y}_{\substack{\frac{\text{dollars}}{\text{year}} \\ \uparrow \\ \text{interest added} \\ \text{to balance}}} - \underbrace{20000}_{\substack{\frac{\text{dollars}}{\text{year}} \\ \uparrow \\ \text{payments removed} \\ \text{from balance}}}}$$

INT. FACTOR

$$\frac{dy}{dt} - 0.05y = -20000 \qquad \mu(t) = e^{\int -0.05\,dt} = e^{-0.05t}$$

$$\frac{d}{dt}\left(e^{-0.05t}y\right) = -20000e^{-0.05t}$$

$$e^{-0.05t}y = 400000e^{-0.05t} + C$$

$$\boxed{y(t) = 400000 + Ce^{0.05t}}$$

(b) The person plans to get a 30-year loan, which mean the balance will be zero in 30 years. What is the initial loan balance, rounded to the nearest dollar?
(This is the home loan amount the person can afford).

$$y(30) = 0 \Rightarrow 0 = 400000 + Ce^{1.5} \Rightarrow C = -\frac{400000}{e^{1.5}}$$

$$y(0) = 400000 - \frac{400000}{e^{1.5}}e^{0} = 400000\left(1 - \frac{1}{e^{1.5}}\right)$$

$$\approx \boxed{\$310,748}$$