Spring 2018 MATH 307 Midterm 1
80 pts total

Name: Heather

Instruction:

- Nothing but writing utensils and a double side 4in $\times$ 6in notecard are allowed.

1. Consider the differential equation

$$\frac{dy}{dt} + y = -t + e^{-t}.$$

[a] (1pts) Is this differential equation linear or nonlinear (no need to explain)?

linear

[b] (17pts) Find the general solution $y(t)$ (Show all steps to get full credit).

It's linear so we can use method of integrating factors

$$I(t) = e^{\int 1\,dt} = e^{t}$$

$$e^{t}(y' + y) = e^{t}(-t + e^{-t})$$

good to check it ✓ $\rightsquigarrow$ ||

$$\frac{d(e^{t}y)}{dt} = -te^{t} + 1$$

$$\int d(e^{t}y) = \int (-te^{t} + 1)\,dt$$

$$e^{t}y = \left(-te^{t} + \int e^{t}\,dt\right) + t$$

$$e^{t}y = -te^{t} + e^{t} + t + C$$

$$y = -t + 1 + \frac{t}{e^{t}} + \frac{C}{e^{t}}$$

$$\boxed{y = \frac{t + C}{e^{t}} + 1 - t}$$

1. (Continued)

[c] (3pts) Find $y(t)$ satisfying the initial value $y(0) = -2$. (Show work to get full credit)

$$-2 = y(0) = C + 1$$

$$\Rightarrow C = -3$$

$$\boxed{y = \frac{t-3}{e^t} + 1 - t}$$

[d] (3pts) For the solution obtained in part [c], what is

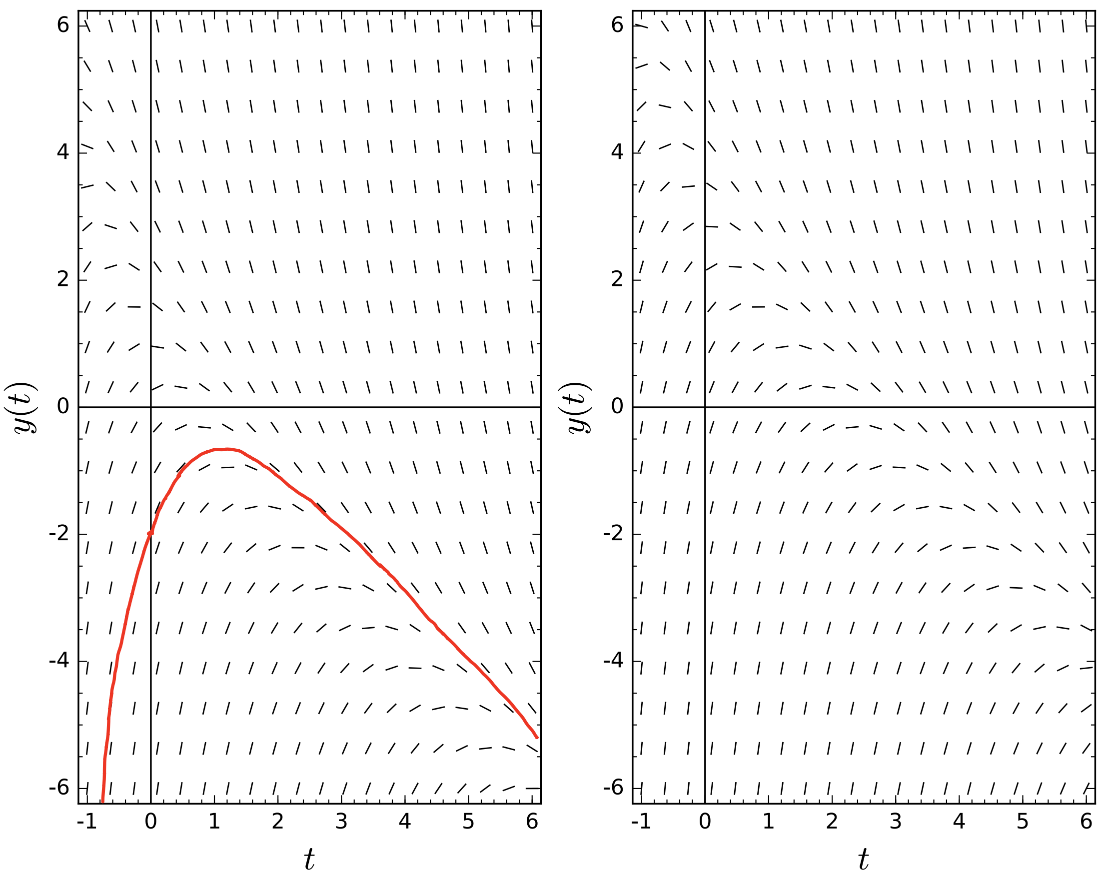
$$\lim_{t\to\infty} y(t) = ?$$

(Explain your reasoning to get full credit.)

$$\lim_{t\to\infty} \underbrace{\frac{t-3}{e^t}}_{\downarrow \atop 0} + 1 - \underbrace{t}_{\downarrow \atop \infty} = \boxed{-\infty}$$

1. (Continued)

[e] (4pts) Which one of the two below is the correct direction field to the differential equation in part [a]? Draw on that direction field the solution curve satisfying the initial condition $y(0) = -2$. (No need to explain.)

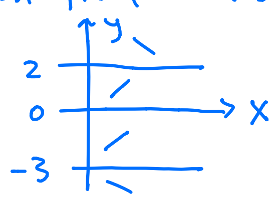


2. [7pts] Write down (no need to show work) a differential equation with the following set of equilibrium solutions and the corresponding stabilities:

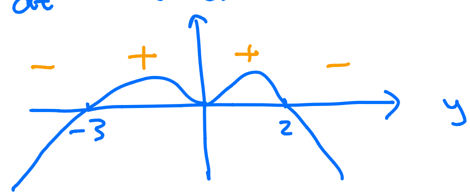
$$y_1 = -3 \text{ (unstable)}, \quad y_2 = 0 \text{ (semistable)}, \quad y_3 = 2 \text{ (stable)}.$$

$$Answer: \boxed{\frac{dy}{dx} = -(y+3)y^2(y-2)}$$

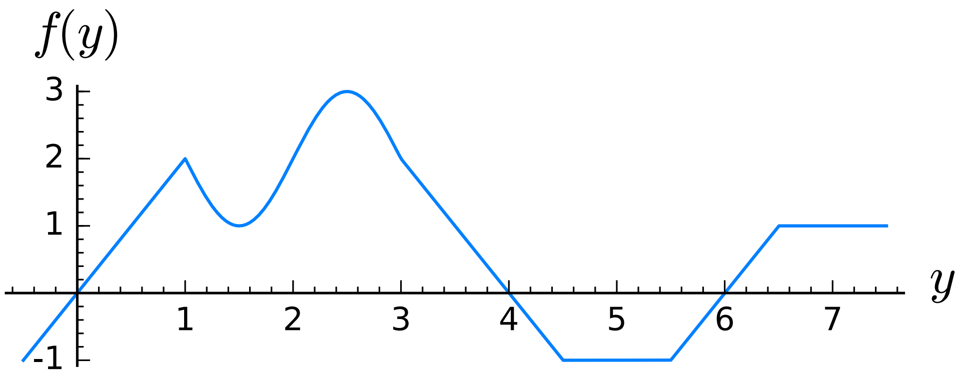
From these info we know direction field looks like :



so $\frac{dy}{dt} = f(y)$ looks like



4. [7pts] Consider the differential equation $\frac{dy}{dx} = f(y)$ where $f(y)$ is sketched in the figure below. Determine the equilibrium solutions, and for each equilibrium solution, classify whether it is stable, unstable, or semistable. (Briefly explain your reasoning to get full credit.)

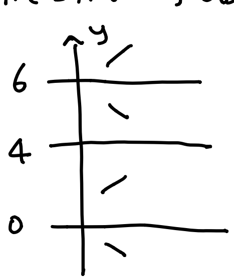


f(y) is − + − +

$f(y) = 0$ when $y = 0, 4, 6$

So equilibrium soln are $y_1 = 0$ (unstable), $y_2 = 4$ (stable), $y_3 = 6$ (unstable)

Direction field looks like



unstable

stable

unstable

4. Genevieve's parents will start saving for her college education as soon as she is born by putting \$40,000 in an index fund that is expected to have a rate of return of about 5% per year. Let $P(t)$ be the amount of money, in dollars, in Genevieve's college fund, with t measured in years. So $P(0) = \$40,000$.

[a] (3pts) Write down a differential equation for $P(t)$ (no need to show work):

$$\frac{dP}{dt} = 0.05 P$$

[b] (4pts) Write down the solution to the above initial value problem (no need to show work).

$$P = 40000\, e^{0.05t}$$

[c] (12pts) To make the numbers easy to calculate, let's suppose that Genevieve will start college at age 20. Use Euler's method with step size of 10 years to get a rough estimate of the amount of money, $P(20)$, in her college fund when she turns 20. (Show all steps to get full credit.)

$$P(0) = 40000$$

$$P'(0) = 0.05\, P(0) = (0.05)(40000) = 2000$$

$$P(10) \approx P(0) + (10)\, P'(0)$$

$$= 40000 + (10)(2000) = 60000$$

$$P'(10) = 0.05\, P(10)$$

$$\approx (0.05)(60000) = 3000$$

$$P(20) \approx P(10) + (10)\, P'(10)$$

$$\approx 60000 + (10)(3000)$$

$$= \boxed{90000}$$

5[a]. (5pts) Let $P(t)$ be the size of a population of fish, with t measured in days. The rate of population increase is proportional to the size of the population. The population increases by 2% per day. Furthermore, 1000 fish are harvested each day. Write down a differential equation for the number of fish in the population. (No need to show work.)

$$\frac{dP}{dt} = 0.02 P - 1000$$

5[b]. (7pts) Max is buying a home and he plans to spend \$2000 per month to pay off a 30-year mortgage. Suppose that the interest rate is 4% compounded continuously. Let $P(t)$ be the amount, in dollars, owed at time t, measured in years. Write down a differential equation for $P(t)$. (No need to show work.)

$$\frac{dP}{dt} = 0.04 P - (2000)(12)$$

6. A tank holds 2000 L of water in which 400 g of salt has been dissolved. Saltwater with a concentration of 2 g/L is pumped in at a rate of 40 L/min. The well mixed salt water solution is pumped out at a rate of 50 L/min. (No need to show work for this problem.)

[a] (2pts) Before the tank becomes empty, what is the total volume, in liters, of water in the tank at time t (measured in minutes)?

$$2000 - 10t$$

[b] (5pts) Let $S(t)$ be the mass (i.e. number of grams) of salt in the tank at time t. Write down a differential equation for $S(t)$.

$$\frac{dS}{dt} = (2)(40) - \left(\frac{S}{2000-10t}\right)(50)$$

$$\frac{dS}{dt} = 80 - \frac{5S}{200-t}$$