Name: ______________________________

Mathematics 307 L
University of Washington

December 10, 2019

FINAL EXAM SOLUTIONS

Here are the rules:

- This exam is closed book. No note sheets, calculators, or electronic devices are allowed.
- In order to receive credit, you must **show all of your work**; to obtain full credit, you must provide mathematical justifications. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Give numerical answers in exact form (for example $\ln(\frac{\pi}{3})$ or $5\sqrt{3}$ or $e^{2.5}$).
- Simplify $e^{a\ln(x)} = x^a$ for $x > 0$.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 9 pages of problems (in addition to the cover sheet and page of formulas). Please make sure that your exam is complete.
- You have 110 minutes to complete the exam.
- HAVE FUN!

Problem	Possible	Score
1	9	
2	9	
3	14	
4	14	
5	21	
6	16	
7	9	
8	8	
Total	100	

Good Luck!

Problem 1. (9 points) Consider the differential equation

$$\left(1+t^2\right)\frac{dy}{dt}+4ty=1$$

(a) (2 points) Circle your answer

(i) Is this a *linear* differential equation? **[YES]** **NO**

(ii) Is this a *separable* differential equation? **YES** **[NO]**

(b) (7 points) Find the general solution, and write your final answer here:

$$y(t) = \frac{t+\frac{1}{3}t^3+C}{(1+t^2)^2}$$

Solution. Rewrite as

$$\frac{dy}{dt}+\frac{4t}{1+t^2}\,y=\frac{1}{1+t^2}$$

$$\mu = e^{\int \frac{4t}{1+t^2}\,dt} = e^{2\log(1+t^2)} = (1+t^2)^2$$

Multiply both sides by μ and simplify the left hand side:

$$\frac{d}{dt}\left((1+t^2)^2\,y\right) = 1+t^2$$

Integrate with indeterminate constant on the right

$$(1+t^2)^2\,y = t+\frac{1}{3}t^3+C$$

Solve for y to get the above answer.

Problem 2. (9 points) A 50 gallon tank initially holds 20 gallons of water that has 4 pounds of chemical Q dissolved in it. Clean water (which does not contain any Q) flows into the tank at the rate of 5 gallons per minute. The mixture flows out of the tank at the rate of 3 gallons per minute. Write the initial value problem for the amount of Q in the tank:

$$\frac{dQ}{dt} = -\frac{3Q}{20+2t} \qquad Q(0) = 4$$

Solve for $Q(t)$, and write the answer here:

$$Q(t) = 4 \cdot (20)^{\frac{3}{2}}(20+2t)^{-\frac{3}{2}}$$

For the first part, use that $V(t) = 20 + 2t$.

Equation is separable (as well as linear, you can use either method).

$$\frac{dQ}{Q} = -\frac{3}{20+2t}\,dt$$
$$\ln|Q| = -\frac{3}{2}\ln|20+2t| + C$$

Since both Q and $20 + 2t$ are positive we can drop the $|\cdot|$ signs.

$$\begin{aligned} Q &= e^{-\frac{3}{2}\ln(20+2t)+C} \\ &= e^{C}e^{-\frac{3}{2}\ln(20+2t)} \\ &= C(20+2t)^{-\frac{3}{2}} \end{aligned}$$

Set $t = 0$ and $Q = 4$ to find $C = 4 \cdot (20)^{\frac{3}{2}}$

Problem 3. (14 points)

Write down a suitable form for $Y(t)$ which would allow you to use the method of undetermined coefficients to find a particular solution to the following equations.

(DO NOT plug Y into the equation or try to find the coefficients.)

(a) (7 points) $y'' + 4y' + 5y = 3t^2e^{-2t} + te^{-2t}\cos(t) + \sin(t) + t + 1$.

Roots of $r^2 + 4r + 5$ are $r = -2 \pm i$. This means the terms with $e^{-2t}\cos(t)$ or $e^{-2t}\sin(t)$ get multiplied by t. There are 4 different types of terms on the right, only one of which gets multiplied by t.

$$y_p(t) = (A_0t^2+A_1t+A_0)e^{-2t}+e^{-2t}\Big(\big(B_1t^2+B_0t\big)\cos(t)+(C_1t^2+C_0t)\sin(t)\Big) + D\cos(t) + E\sin(t) + F_1t + F_0.$$

(b) (7 points) $y'' + 2y' - 3y = 3t^2e^{-3t} + te^t\sin(2t) + 7e^t + t + 1$.

Roots of $r^2 + 2r - 3$ are $r = 1, -3$. This means the terms with e^{-3t} or e^t get multiplied by t. There are 4 different types of terms on the right, two of which gets multiplied by t.

$$y_p(t) = (A_2t^3+A_1t^2+A_0t)e^{-3t}+e^t\Big(\big(B_1t+B_0\big)\cos(2t)+(C_1t+C_0)\sin(2t)\Big) + Dte^t + F_1t + F_0.$$

Problem 4. (14 points) Consider the initial value problem

$$u'' + u' + \frac{5}{4}u = 0\,, \qquad u(0) = -2\ , \quad u'(0) = -1.$$

(a) (7 points) Solve the initial value problem, using whichever method you prefer.

$$u(t) = e^{-\frac{1}{2}t}\big(-2\cos(t) - 2\sin(t)\big)$$

(b) (4 points) Express the answer in the form $u(t) = Ae^{\rho t}\cos(\omega t - \phi)$, where $A > 0$.

$A = 2\sqrt{2}$ $\quad \rho = -\frac{1}{2}$ $\quad \omega = 1$ $\quad \phi = \frac{5\pi}{4}$ or $-\frac{3\pi}{4}$

(c) (3 points) Find the first time $t > 0$ at which $u(t) = 0$.

$$t = \frac{3\pi}{4}$$

The general solution is $y = e^{-\frac{1}{2}t}\big(c_1\cos(t) + c_2\sin(t)\big)$. Plug in initial conditions and solve to get $c_1 = -2$ and $c_2 = -2$.

$A = \sqrt{c_1^2 + c_2^2} = \sqrt{8} = 2\sqrt{2}$. $\tan\phi = \frac{-2}{-2} = 1$. However, the point $(-2, -2)$ is in the third quadrant, so we take $\phi = \tan^{-1}(1) \pm \pi$.

$u(t) = 0$ when $\sin(t) = -\cos(t)$ so $\tan(t) = -1$. The first value of $t > 0$ for which this happens is $t = \dfrac{3\pi}{4}$.

Problem 5. (21 points) For each of the following functions $F(s)$, find $f(t) = \mathcal{L}^{-1}[F(s)]$. That is, find $f(t)$ for which the Laplace transform of f satisfies $\mathcal{L}[f(t)] = F(s)$.

For answers that involve Heaviside functions, express your answers in piece-wise form (i.e. give the formula for $f(t)$ on the different intervals).

Draw a box around each of your answers.

(a) (7 points) $F(s) = \dfrac{1}{s-1} + \dfrac{e^{-3s}}{s^2}$

$y(t) = e^t + u_3(t)(t-3)$, so in piecewise form

$$\boxed{y(t) = \begin{cases} e^t, & t \leq 3 \\ e^t + (t-3), & t > 3 \end{cases}}$$

(Parts (b) and (c) of this problem are on the next page.)

(b) (7 points) $F(s) = \dfrac{1}{s\left(s^2+s+\frac{1}{2}\right)}$

Partial fractions:

$$\begin{aligned}\frac{1}{s\left(s^2+s+\frac{1}{2}\right)} &= \frac{2}{s} + \frac{-2s-2}{s^2+s+\frac{1}{2}} \\ &= \frac{2}{s} - 2\,\frac{(s+\frac{1}{2})+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{1}{2})^2}\end{aligned}$$

$$\boxed{y(t) = 2 - 2e^{-\frac{1}{2}t}\cos(\tfrac{1}{2}t) - 2e^{-\frac{1}{2}t}\sin(\tfrac{1}{2}t)}$$

(c) (7 points) $F(s) = \left(1-e^{-2s}\right)\left(\dfrac{1-s}{s^2} + \dfrac{s+1}{s^2+2s+2}\right)$

Let $G(s) = \dfrac{1-s}{s^2} + \dfrac{s+1}{s^2+2s+2} = \dfrac{1}{s^2} - \dfrac{1}{s} + \dfrac{s+1}{(s+1)^2+1}$

This is the Laplace transform of $g(t) = t - 1 + e^{-t}\cos(t)$.

Then $f(t) = g(t) - u_2(t)g(t-2)$. In piecewise form

$$\boxed{y(t) = \begin{cases} t-1+e^{-t}\cos(t), & t \le 2 \\ t-1+e^{-t}\cos(t) - \Big((t-2)-1+e^{-(t-2)}\cos(t-2)\Big), & t > 2 \end{cases}}$$

Problem 6. (16 points) For the following problems find $Y(s)$, where $Y(s) = \mathcal{L}[y(t)]$ is the Laplace transform of the solution to the equation. (DO NOT solve the equation or find $y(t)$, just $Y(s)$. You do not need to apply partial fractions to simplify $Y(s)$.) **Draw a box around each of your answers**.

(a) (8 points) $y''+4y'+3y = f(t)\,, \quad y(0) = 1\,, \quad y'(0) = 0\,, \quad f(t) = \begin{cases} t\,, & 0 \le t < 1 \\ 0\,, & 1 \le t < \infty \end{cases}$

Write $f(t) = (1 - u_1(t))\,t = t - u_1(t)(t-1) - u_1(t)$, so

$$\boxed{\mathcal{L}(f) = F(s) = \frac{1}{s^2} - e^{-t}\Big(\frac{1}{s^2} - \frac{1}{s}\Big)}$$

$$(s^2 + 4s + 3)Y(s) = s + 4 + F(s)$$

$$\boxed{Y(s) = \frac{s+4}{s^2+4s+3} + \frac{1}{s^2+4s+3}\bigg(\frac{1}{s^2} - e^{-t}\Big(\frac{1}{s^2} - \frac{1}{s}\Big)\bigg)}$$

(b) (8 points) $y''+6y = f(t)\,, \quad y(0) = 1\,, \quad y'(0) = 2\,, \quad f(t) = \begin{cases} 0\,, & 0 \le t < \pi \\ \sin t\,, & \pi \le t < 2\pi \\ 0\,, & 2\pi \le t < \infty \end{cases}$

$f(t) = \big(u_\pi(t) - u_{2\pi}(t)\big)\sin t$.

So $\mathcal{L}(f) = F(s) = e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\} - e^{-2\pi s}\mathcal{L}\{\sin(t+2\pi)\}$

Use that $\sin(t+\pi) = -\sin(t)$ and $\sin(t+2\pi) = \sin(t)$ to find

$$\boxed{F(s) = -\big(e^{-\pi s} + e^{-2\pi s}\big)\frac{\pi}{s^2+\pi^2}}$$

From the equation,

$$(s^2 + 6)Y(s) = s + 2 + F(s)$$

$$\boxed{Y(s) = \frac{s+2}{s^2+6} - \frac{\pi\big(e^{-\pi s} + e^{-2\pi s}\big)}{(s^2+6)(s^2+\pi^2)}}$$

Problem 7. (9 points) Solve the following differential equation

$$y'' + 25y = 1 + 3\delta_4(t), \qquad y(0) = 1\,, \quad y'(0) = 1.$$

You may use Heaviside functions in your answer.

$$y(t) = \frac{24}{25}\cos(5t) + \frac{1}{5}\sin(5t) + \frac{1}{25} + \frac{3}{5}u_4(t)\sin\big(5(t-4)\big)$$

$(s^2 + 25)Y(s) = s + 1 + \dfrac{1}{s} + 3e^{-4t}$, so

$$Y(s) = \frac{s+1}{s^2+25} + \frac{1}{s(s^2+25)} + 3e^{-4t}\frac{1}{s^2+25}$$

Use partial fractions to write

$$\frac{1}{s(s^2+25)} = \frac{1}{25}\Big(\frac{1}{s} - \frac{s}{s^2+25}\Big)$$

so

$$Y(s) = \frac{24}{25}\,\frac{s}{s^2+5^2} + \frac{1}{5}\,\frac{5}{s^2+25} + \frac{1}{25}\,\frac{1}{s} + \frac{3}{5}\,e^{-4t}\frac{5}{s^2+25}$$

so $y(t) = \dfrac{24}{25}\cos(5t) + \dfrac{1}{5}\sin(5t) + \dfrac{1}{25} + \dfrac{3}{5}u_4(t)\sin\big(5(t-4)\big)$

Problem 8. (8 points) A mass of 1 kg is hanging on a spring of spring constant $k = 4\,\text{N/m}$. The system is at rest at its equilibrium point at time $t = 0$. There is no damping.

You apply a downward force of 1 N to the mass for T seconds, after which you turn off the force, so that the solution is in a steady state for $t > T$. Find the amplitude of this steady state. (Your answer will be a function of T.)

$$A = \frac{1}{4}\sqrt{(1-\cos(2T))^2 + \sin(2T)^2} = \tfrac{1}{4}\sqrt{2 - 2\cos(2T)}$$

The driving force is $f(t) = -(1 - u_T(t))$, so $\mathcal{L}\{f\} = F(s) = \dfrac{-1 + e^{-Ts}}{s}$.

The equation leads to

$$\mathcal{L}\{y\} = Y(s) = \left(-1 + e^{-Ts}\right)\frac{1}{s(s^2+4)}$$

Partial fractions

$$\frac{1}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) = \mathcal{L}\left\{\frac{1}{4} - \frac{1}{4}\cos(2t)\right\}$$

Therefore

$$y(t) = -\frac{1}{4} + \frac{1}{4}\cos(2t) + u_T(t)\left(\frac{1}{4} - \frac{1}{4}\cos\bigl(2(t-T)\bigr)\right)$$

For $t > T$ we have

$$y(t) = \frac{1}{4}\Bigl(\cos(2t) - \cos(2t - 2T)\Bigr)$$

To find the amplitude we expand

$$\cos(2t - 2T) = \cos(2T)\cos(2t) + \sin(2T)\sin(2t)$$

to write, for $t > T$,

$$y(t) = \frac{1 - \cos(2T)}{4}\cos(2t) - \frac{\sin(2T)}{4}\sin(2t)$$

and the amplitude is as given above.