Name: ______________________________

Mathematics 307 L
University of Washington

December 10, 2019

FINAL EXAM

Here are the rules:

- This exam is closed book. No note sheets, calculators, or electronic devices are allowed.
- In order to receive credit, you must **show all of your work**; to obtain full credit, you must provide mathematical justifications. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Give numerical answers in exact form (for example $\ln(\frac{\pi}{3})$ or $5\sqrt{3}$ or $e^{2.5}$).
- Simplify $e^{a\ln(x)} = x^a$ for $x > 0$.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 9 pages of problems (in addition to the cover sheet and page of formulas). Please make sure that your exam is complete.
- You have 110 minutes to complete the exam.
- HAVE FUN!

Problem	Possible	Score
1	9	
2	9	
3	14	
4	14	
5	21	
6	16	
7	9	
8	8	
Total	100	

Good Luck!

Problem 1. (9 points) Consider the differential equation

$$\left(1+t^2\right)\frac{dy}{dt}+4ty=1$$

(a) (2 points) Circle your answer

(i) Is this a *linear* differential equation? **YES** **NO**

(ii) Is this a *separable* differential equation? **YES** **NO**

(b) (7 points) Find the general solution, and write your final answer here:

$y(t) =$

Problem 2. (9 points) A 50 gallon tank initially holds 20 gallons of water that has 4 pounds of chemical Q dissolved in it. Clean water (which does not contain any Q) flows into the tank at the rate of 5 gallons per minute. The mixture flows out of the tank at the rate of 3 gallons per minute. Write the initial value problem for the amount of Q in the tank:

$\frac{dQ}{dt} =$	$Q(0) =$

Solve for $Q(t)$, and write the answer here:

$Q(t) =$

Problem 3. (14 points)

Write down a suitable form for $Y(t)$ which would allow you to use the method of undetermined coefficients to find a particular solution to the following equations.

(DO NOT plug Y into the equation or try to find the coefficients.)

(a) (7 points) $$y'' + 4y' + 5y = 3t^2 e^{-2t} + te^{-2t}\cos(t) + \sin(t) + t + 1\,.$$

(b) (7 points) $$y'' + 2y' - 3y = 3t^2 e^{-3t} + te^{t}\sin(2t) + 7e^{t} + t + 1\,.$$

Problem 4. (14 points) Consider the initial value problem

$$u'' + u' + \frac{5}{4}\,u = 0\,, \qquad u(0) = -2\,, \quad u'(0) = -1.$$

(a) (7 points) Solve the initial value problem, using whichever method you prefer.

$u(t) =$

(b) (4 points) Express the answer in the form $u(t) = Ae^{\rho t}\cos(\omega t - \phi)$, where $A > 0$.

$A =$	$\rho =$	$\omega =$	$\phi =$

(c) (3 points) Find the first time $t > 0$ at which $u(t) = 0$.

$t =$

Problem 5. (21 points) For each of the following functions $F(s)$, find $f(t) = \mathcal{L}^{-1}[F(s)]$. That is, find $f(t)$ for which the Laplace transform of f satisfies $\mathcal{L}[f(t)] = F(s)$.

For answers that involve Heaviside functions, express your answers in piece-wise form (i.e. give the formula for $f(t)$ on the different intervals).

Draw a box around each of your answers.

(a) (7 points) $F(s) = \dfrac{1}{s-1} + \dfrac{e^{-3s}}{s^2}$

(Parts (b) and (c) of this problem are on the next page.)

(b) (7 points) $F(s) = \dfrac{1}{s\left(s^2 + s + \frac{1}{2}\right)}$

(c) (7 points) $F(s) = \left(1 - e^{-2s}\right)\left(\dfrac{1-s}{s^2} + \dfrac{s+1}{s^2+2s+2}\right)$

Problem 6. (16 points) For the following problems find $Y(s)$, where $Y(s) = \mathcal{L}[y(t)]$ is the Laplace transform of the solution to the equation. (DO NOT solve the equation or find $y(t)$, just $Y(s)$. You do not need to apply partial fractions to simplify $Y(s)$.) **Draw a box around each of your answers**.

(a) (8 points) $y''+4y'+3y = f(t)\,, \quad y(0) = 1\,, \quad y'(0) = 0\,, \quad f(t) = \begin{cases} t\,, & 0 \le t < 1 \\ 0\,, & 1 \le t < \infty \end{cases}$

(b) (8 points) $y''+6y = f(t)\,, \qquad y(0) = 1\,, \quad y'(0) = 2\,, \qquad f(t) = \begin{cases} 0\,, & 0 \le t < \pi \\ \sin t\,, & \pi \le t < 2\pi \\ 0\,, & 2\pi \le t < \infty \end{cases}$

Problem 7. (9 points) Solve the following differential equation

$$y'' + 25y = 1 + 3\delta_4(t), \qquad y(0) = 1\,, \quad y'(0) = 1.$$

You may use Heaviside functions in your answer.

$y(t) =$

Problem 8. (8 points) A mass of 1 kg is hanging on a spring of spring constant $k = 4\,\mathrm{N/m}$. The system is at rest at its equilibrium point at time $t = 0$. There is no damping.

You apply a downward force of 1 N to the mass for T seconds, after which you turn off the force, so that the solution is in a steady state for $t > T$. Find the amplitude of this steady state. (Your answer will be a function of T.)

$A =$

Submitted by Name: ____ *on December 10, 2019.*