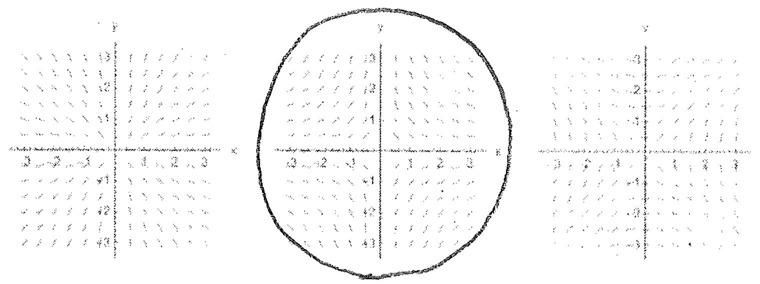
Q1 (10 points)

This problem concerns the differential equation $\frac{dy}{dx} = -\frac{y}{x}$.

(a) **Circle** the correct direction field for this equation.



(b) Find an explicit solution $y(x)$ satisfying the condition $y(1) = 2$.

$$\int \frac{1}{y}\,dy = \int -\frac{1}{x}\,dx$$

$$\ln|y| = -\ln|x| + C$$

$$|y| = e^{-\ln|x|} e^{C}$$

$$|y| = A|x|^{-1}$$

$$y = \frac{A}{x}$$

$$y(1) = 2 \Rightarrow A = 2$$

$$\boxed{y = \frac{2}{x}}$$

Q2 (10 points)

Solve the following (related!) problems.

(a) Solve the initial value problem

$$y'' + 2y' + 5y = \delta(t), \qquad y(0) = y'(0) = 0$$

$\downarrow \mathcal{L}$

$$s^2 Y + 2sY + 5Y = 1$$

$$Y(s) = \frac{1}{s^2 + 2s + 5} = \frac{1}{(s+1)^2 + 4}$$

$$\boxed{y(t) = \frac{1}{2} e^{-t} \sin 2t}$$

(b) Express the solution to the initial value problem

$$y'' + 2y' + 5y = \sin(t^2), \qquad y(0) = y'(0) = 0$$

as an **integral**.

$$Y(s) = \frac{1}{s^2 + 2s + 5} \mathcal{L}(\sin t^2)$$

$$y(t) = \left(\frac{1}{2} e^{-t} \sin 2t\right) * (\sin t^2)$$

$$= \boxed{\int_0^t \frac{1}{2} e^{-(t-u)} \sin 2(t-u) \sin u^2 \, du}$$

Q3 (10 points)

Find the **inverse Laplace transforms** of the following functions:

(a) $F(s) = \dfrac{2s+1}{s^2+2s+1}$

$$= \frac{2(s+1)-1}{(s+1)^2} = \frac{2}{s+1} - \frac{1}{(s+1)^2}$$

$$\boxed{f(t) = 2e^{-t} - te^{-t}}$$

(b) $F(s) = \dfrac{1-e^{-4s}}{s^3+s^2}$

$$= (1-e^{-4s}) \cdot \frac{1}{s^2(s+1)}$$

$$\frac{1}{s^2(s+1)} = \frac{As+B}{s^2} + \frac{C}{s+1}$$

$$= (1-e^{-4s})\left(\frac{-s+1}{s^2} + \frac{1}{s+1}\right)$$

$$1 = As^2 + Bs + As + B + Cs^2$$

$$= (1-e^{-4s})\left(-\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right)$$

$0 = A + C$
$0 = B + A$
$1 = B$

$A = -1$
$C = 1$

$$= -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1} - e^{-4s}\left(-\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right)$$

$$\boxed{f(t) = -1 + t + e^{-t} - u_4(t)\cdot\left(-1 + (t-4) + e^{-(t-4)}\right)}$$

Q4 (10 points)

(a) The linear 2nd-order homogeneous equation

$$y'' - \frac{2}{t^2}y = 0$$

doesn't have constant coefficients, so our usual guess of $y(t) = e^{rt}$ won't work. Instead, this equation has solutions of the form $\underbrace{y(t) = t^r}$. Find two linearly independent solutions of this form.

idea: plug this into the differential equation

$y = t^r$

$y' = rt^{r-1}$

$y'' = r(r-1)t^{r-2}$

$r(r-1)t^{r-2} - \frac{2}{t^2}\cdot t^r = 0$

$r(r-1) - 2 = 0$

$r^2 - r - 2 = 0$

$(r+1)(r-2) = 0$

$r = -1, 2$

$\boxed{y = t^{-1}, \quad y = t^2}$

(b) The 2nd-order **inhomogeneous** equation

$$y'' - \frac{2}{t^2}y = 1$$

has a particular solution of the surprising form $y_p(t) = At^2 \ln t$. Find the value of A.

$y_p = At^2 \ln t$

$y_p' = 2At \ln t + At$

$y_p'' = 2A \ln t + 3A$

$\cancel{2A\ln t} + 3A - \frac{2}{\cancel{t^2}} \cancel{At^2 \ln t} = 1$

$3A = 1, \quad \boxed{A = \frac{1}{3}}$

(c) What is the general solution to the equation in part (b)? [Hint: you've already done most of the work above!]

$$\boxed{y(t) = C_1 t^{-1} + C_2 t^2 + \frac{1}{3}t^2 \ln t}$$

Q5 (10 points)

Solve the initial value problem

$$y'' + y = \begin{cases} 1 & \text{if } 0 \le t < 3 \\ 0 & \text{if } 3 \le t \end{cases} \qquad y(0) = 0, \quad y'(0) = 1$$

$\downarrow \mathcal{L}$ $\qquad = 1 - u_3$

$$s^2 Y - 1 + Y = \frac{1}{s} - \frac{e^{-3s}}{s}$$

$$Y(s) \cdot (s^2+1) = \frac{1-e^{-3s}}{s} + 1$$

$$Y(s) = \frac{1-e^{-3s}}{s(s^2+1)} + \frac{1}{s^2+1}$$

$$= (1-e^{-3s}) \cdot \frac{1}{s(s^2+1)} + \frac{1}{s^2+1}$$

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$1 = As^2 + A + Bs^2 + Cs$$

$A = 1$

$B = -1$

$C = 0$

$$= (1-e^{-3s}) \left(\frac{1}{s} - \frac{s}{s^2+1} \right) + \frac{1}{s^2+1}$$

$$= \frac{1}{s} - \frac{s}{s^2+1} - e^{-3s} \left(\frac{1}{s} - \frac{s}{s^2+1} \right) + \frac{1}{s^2+1}$$

$$\boxed{y(t) = 1 - \cos t - u_3(t) \cdot (1 - \cos(t-3)) + \sin t}$$

Q6 (10 points)

(a) Find the Laplace transform of

$$f(t) = \begin{cases} 2-t & \text{if } 0 \le t < 2 \\ t^2 - 4t + 4 & \text{if } 2 \le t \end{cases}$$

$$= (1-u_2)\cdot(2-t) + u_2\cdot(t^2-4t+4)$$

$$= 2-t + u_2(t)\cdot(t-2) + u_2(t)\cdot(t-2)^2$$

$$F(s) = \boxed{\frac{2}{s} - \frac{1}{s^2} + \frac{1}{s^2}e^{-2s} + \frac{2}{s^3}e^{-2s}}$$

(b) The function $1/\sqrt{t}$ has a rather interesting Laplace transform:

$$\mathcal{L}\left(\frac{1}{\sqrt{t}}\right) = \sqrt{\frac{\pi}{s}}.$$

Using this fact, find the Laplace transform of $f(t) = \dfrac{e^t + e^{3t}}{\sqrt{t}}$.

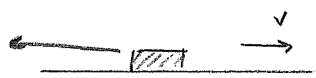
$$f(t) = e^t \cdot \frac{1}{\sqrt{t}} + e^{3t} \cdot \frac{1}{\sqrt{t}}$$

$$F(s) = \boxed{\sqrt{\frac{\pi}{s-1}} + \sqrt{\frac{\pi}{s-3}}}$$ (by the exponential shift rule)

Q7 (10 points)

A rocket sled with mass 500 kg and initial velocity 100 m/s is slowed by a channel of water. Suppose the channel exerts a force proportional to the **square** of the sled's velocity, acting in the opposite direction. You can assume there are no other forces acting on the sled.

If it takes just 1 second to slow the sled to a velocity of 50 m/s, when will its velocity reach 10 m/s?



$$F_{water} = -kv^2$$

$$500v' = -kv^2 \quad \text{(Newton's 2nd Law)}$$

$$\int \frac{1}{v^2}\,dv = \int -\frac{k}{500}\,dt$$

$$-\frac{1}{v} = -\frac{kt}{500} + C$$

$$v(0) = 100 \Rightarrow C = -\frac{1}{100}$$

$$\frac{1}{v} = \frac{kt}{500} + \frac{1}{100}$$

$$\frac{1}{v} = \frac{kt+5}{500}$$

$$v = \frac{500}{kt+5}$$

$$v(1) = 50 \Rightarrow k = 5$$

$$v = \frac{100}{t+1}$$

$$10 = \frac{100}{t+1}$$

$$\boxed{t = 9}$$

Q8 (10 points)

A block of unknown mass m (measured in kg) is attached to a spring with unknown spring constant k.

The system also contains a variable damping mechanism that can be turned on and off.

First suppose the damping is turned off (so $\gamma = 0$ initially). When the block is pushed from equilibrium position with an initial velocity of 1 m/sec, the block oscillates with an **amplitude** of 2 meters.

Next, the damping coefficient γ is increased until the precise value where oscillations no longer appear. **With what initial velocity** should the block now be pushed from equilibrium position in order to attain the same maximum distance of 2 meters from equilibrium position? Give your final answer as a number (in particular, it should not involve m or k).

Critically damped! $\gamma^2 = 4mk$

Part 1 (no damping): $my'' + ky = 0$

$y(t) = C_1 \cos\omega t + C_2 \sin\omega t, \quad \omega = \sqrt{\frac{k}{m}}$

$y(0) = 0 \Rightarrow C_1 = 0$

$y(t) = \underbrace{C_2}_{2\text{, since the amplitude is } 2} \sin\omega t$

$y'(t) = 2\omega \cos\omega t$

$y'(0) = 1 \Rightarrow \omega = \frac{1}{2}$, so $\sqrt{\frac{k}{m}} = \frac{1}{2}$ (we will use this in the 2nd part!)

Part 2 (critically damped): $my'' + \gamma y' + ky = 0, \quad \gamma^2 = 4mk$

$y(t) = C_1 e^{-\frac{\gamma}{2m}t} + C_2 t e^{-\frac{\gamma}{2m}t}$

$y(0) = 0 \Rightarrow C_1 = 0$

$y(t) = C_2 t e^{-\frac{\gamma}{2m}t} = C_2 t e^{-\frac{\sqrt{4mk}}{2m}t} = C_2 t e^{-\sqrt{\frac{k}{m}}t} = C_2 t e^{-\frac{1}{2}t}$

(using $\gamma^2 = 4mk$) (using $\sqrt{\frac{k}{m}} = \frac{1}{2}$)

$y'(t) = C_2 e^{-\frac{1}{2}t} - \frac{1}{2} C_2 t e^{-\frac{1}{2}t}$

Set $0 = C_2 e^{-\frac{1}{2}t}\left(1 - \frac{1}{2}t\right)$, so the maximum occurs at $t = 2$.

We want $y(2) = 2$, so $2 = 2C_2 e^{-1}$, or $C_2 = e$,

But $y'(0) = C_2$, so C_2 is also the initial velocity.

Therefore $v_0 = \boxed{e}$. A very difficult problem!