MATH 207G: FINAL EXAM

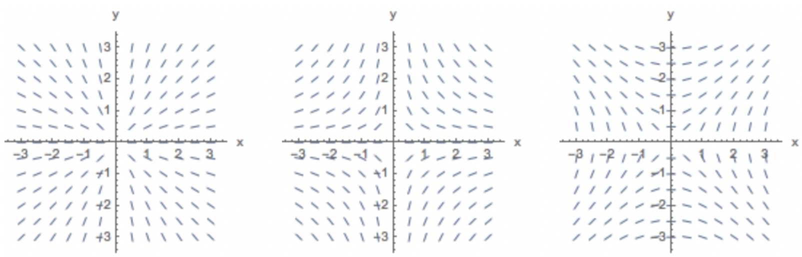
Name: ______________________________

Please do **not** start working until given the indication. You have 1 hour and 50 minutes for the exam, which has 8 problems. Good luck!

Q1 (10 points)

This problem concerns the differential equation $\frac{dy}{dx} = -\frac{y}{x}$.

(a) **Circle** the correct direction field for this equation.



(b) Find an explicit solution $y(x)$ satisfying the condition $y(1) = 2$.

Q2 (10 points)

Solve the following (related!) problems.

(a) Solve the initial value problem

$$y'' + 2y' + 5y = \delta(t), \qquad y(0) = y'(0) = 0$$

(b) Express the solution to the initial value problem

$$y'' + 2y' + 5y = \sin(t^2), \qquad y(0) = y'(0) = 0$$

as an **integral**.

Q3 (10 points)

Find the **inverse Laplace transforms** of the following functions:

(a) $F(s) = \dfrac{2s+1}{s^2+2s+1}$

(b) $F(s) = \dfrac{1-e^{-4s}}{s^3+s^2}$

Q4 (10 points)

(a) The linear 2nd-order homogeneous equation

$$y'' - \frac{2}{t^2} y = 0$$

doesn't have constant coefficients, so our usual guess of $y(t) = e^{rt}$ won't work. Instead, this equation has solutions of the form $y(t) = t^r$. Find two linearly independent solutions of this form.

(b) The 2nd-order **inhomogeneous** equation

$$y'' - \frac{2}{t^2} y = 1$$

has a particular solution of the surprising form $y_p(t) = At^2 \ln t$. Find the value of A.

(c) What is the general solution to the equation in part (b)? [Hint: you've already done most of the work above!]

Q5 (10 points)

Solve the initial value problem

$$y'' + y = \begin{cases} 1 & \text{if } 0 \leq t < 3 \\ 0 & \text{if } 3 \leq t \end{cases} \qquad y(0) = 0, \quad y'(0) = 1$$

Q6 (10 points)

(a) Find the Laplace transform of

$$f(t) = \begin{cases} 2 - t & \text{if } 0 \leq t < 2 \\ t^2 - 4t + 4 & \text{if } 2 \leq t \end{cases}$$

(b) The function $1/\sqrt{t}$ has a rather interesting Laplace transform:

$$\mathcal{L}\left(\frac{1}{\sqrt{t}}\right) = \sqrt{\frac{\pi}{s}}.$$

Using this fact, find the Laplace transform of $f(t) = \dfrac{e^t + e^{3t}}{\sqrt{t}}$.

Q7 (10 points)

A rocket sled with mass 500 kg and initial velocity 100 m/s is slowed by a channel of water. Suppose the channel exerts a force proportional to the **square** of the sled's velocity, acting in the opposite direction. You can assume there are no other forces acting on the sled.

If it takes just 1 second to slow the sled to a velocity of 50 m/s, when will its velocity reach 10 m/s?

Q8 (10 points)

A block of unknown mass m (measured in kg) is attached to a spring with unknown spring constant k.

The system also contains a variable damping mechanism that can be turned on and off.

First suppose the damping is turned off (so $\gamma = 0$ initially). When the block is pushed from equilibrium position with an initial velocity of 1 m/sec, the block oscillates with an **amplitude** of 2 meters.

Next, the damping coefficient γ is increased until the precise value where oscillations no longer appear. **With what initial velocity** should the block now be pushed from equilibrium position in order to attain the same maximum distance of 2 meters from equilibrium position? Give your final answer as a number (in particular, it should not involve m or k).