Math 307Q **Final Exam A** **Winter 2018**

Your Name

Your Signature

Section (circle one) MA MB MC

Problem	Total Points	Score
1	15	
2	15	
3	20	
4	15	
5	15	
6	20	
Total	100	

- This exam is closed book. No Notes. If you forget a formula, ask one of us.
- No graphing or symbolic calculators are allowed. You may not use cell phones during the exam.
- Show your work. Do not do computations in your head. Instead, write them out on the exam paper.
- **Place a box around $\boxed{\textbf{YOUR FINAL ANSWER}}$ to each question.**
- If you need more room, use the backs of the pages and indicate that you have done so.
- If you are not sure what a question means, raise your hand and ask us.
- The hints are suggestions only.

$\boxed{1}$ (15 points) Find the Laplace Transform of each function:

(a) $f(t) = t^2 e^{3t}$

$$\mathcal{L}\{t^2 e^{3t}\} = \mathcal{L}\{t^2\}\Big|_{s \to s-3}$$

$$= \frac{2}{s^3}\Big|_{s \to s-3} = \frac{2}{(s-3)^3}$$

(b) $f(t) = \begin{cases} 0 & 0 < t < 4 \\ e^{-3t} & 4 \le t \end{cases} = e^{-3t} u_4(t)$

$$\mathcal{L}\{u_4(t) e^{-3t}\} = e^{-4s} \mathcal{L}\{e^{-3(t+4)}\}$$

$$= e^{-4s} e^{-12} \mathcal{L}\{e^{-3t}\}$$

$$= e^{-4s} e^{-12} \frac{1}{s+3}$$

(c) $f(t) = \begin{cases} \sin(2t) & 0 < t < \frac{\pi}{4} \\ 0 & \frac{\pi}{4} \le t \end{cases} = (1 - u_{\frac{\pi}{4}}(t)) \sin 2t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin 2t\} - \mathcal{L}\{u_{\frac{\pi}{4}}(t) \sin 2t\}$$

$$= \frac{2}{s^2+4} - e^{-\frac{\pi}{4}s} \mathcal{L}\{\sin 2(t + \tfrac{\pi}{4})\}$$

$$- e^{-\frac{\pi}{4}s} \mathcal{L}\{\sin(2t + \tfrac{\pi}{2})\}$$

$$- e^{-\frac{\pi}{4}s} \mathcal{L}\{\cos(2t)\}$$

$$= \frac{2}{s^2+4} - e^{-\frac{\pi}{4}s} \frac{s}{s^2+4}$$

$\boxed{2}$ (15 points) Find the inverse Laplace Transforms.

(a) $\frac{1}{s^2+2s} = \frac{A}{s} + \frac{B}{s+2} = \frac{\frac{1}{2}}{s} - \frac{\frac{1}{2}}{s+2}$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s}\right\} = \frac{1}{2} - \frac{1}{2}e^{-2t}$

(b) $\frac{s}{s^2+4s+13} = \frac{s}{(s+2)^2+9} = \frac{s+2}{(s+2)^2+9} - \frac{2}{(s+2)^2+9}$

$\mathcal{L}^{-1}\left\{\frac{s}{(s+2)^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}e^{-2t} - \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}e^{-2t}$

$= \cos 3t \cdot e^{-2t} - \frac{2}{3}\sin 3t \cdot e^{-2t}$

(c) $\frac{e^{-s}}{(s+1)^2+1}$ $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+1)^2+1}\right\} = u_1(t)\,\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}\Big|_{t\to t-1}$

$= u_1(t)\left(e^{-t}\sin t\Big|_{t\to t-1}\right)$

$= u_1(t)\, e^{-(t-1)}\sin(t-1)$

$\boxed{3}$ (20 points)

(a) Solve **using Laplace transforms**.

$$y'' + 5y' + 6y = 0$$
$$y(0) = 1 \qquad y'(0) = 2$$

$$s^2Y - 1s - 2 + 5(sY - 1) + 6Y = 0$$

$$(s^2 + 5s + 6)Y = s + 7$$

$$Y(s) = \frac{s+7}{(s+3)(s+2)} = \frac{-4}{s+3} + \frac{5}{s+2}$$

$$y(t) = -4e^{-3t} + 5e^{-2t}$$

(b) Find a formula for the solution as a convolution integral.

$$y'' + 5y' + 6y = e^{-3t^2}$$
$$y(0) = 0 \qquad y'(0) = 0$$

$$G(s) = \frac{1}{(s+3)(s+2)} = \frac{1}{s+2} - \frac{1}{s+3}$$

Impulse Respone $g(t) = e^{-2t} - e^{-3t}$

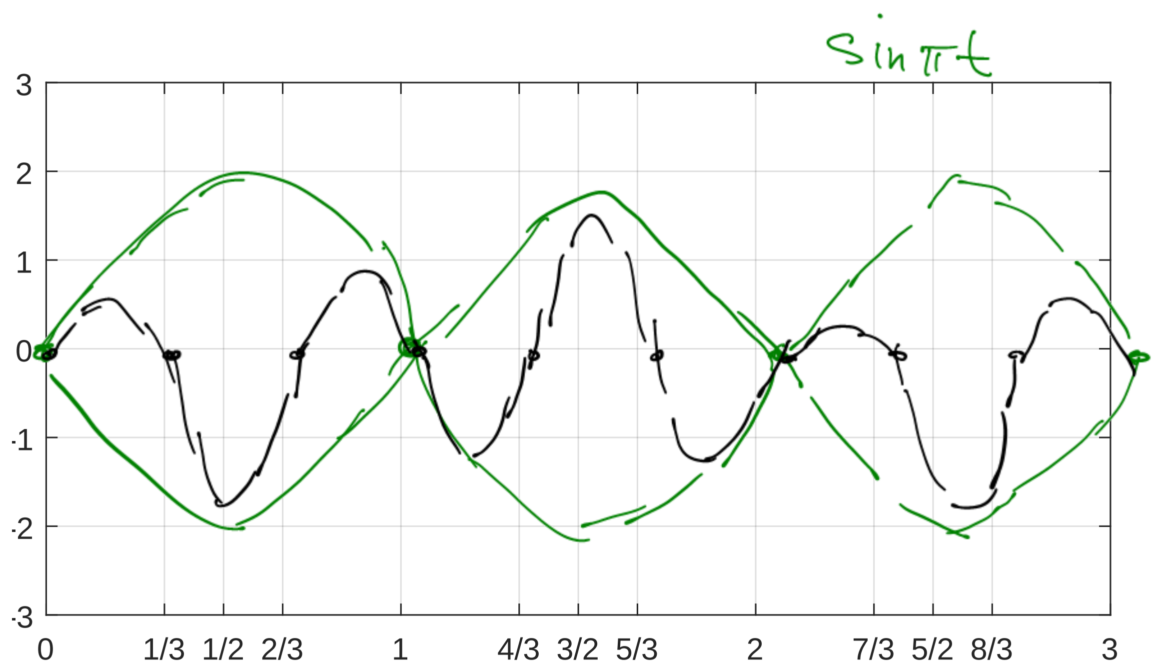
$$y(t) = \int_0^t \left(e^{-2(t-\tau)} - e^{-3(t-\tau)}\right) e^{-3\tau^2}\, d\tau$$

$\boxed{4}$ (15 points) Write $y(t) = \cos(2\pi t) - \cos(4\pi t)$ as a product and sketch its graph. *The positions of the grid lines have been chosen to help you draw the graph.*

$\cos(3\pi t - \pi t) = \cos 3\pi t \cos \pi t + \sin 3\pi t \sin \pi t$

$- \quad \cos(3\pi t + \pi t) = \cos 3\pi t \cos \pi t - \sin 3\pi t \sin \pi t$

$= 2 \sin 3\pi t \sin \pi t$



Write a second order intial value problem that $y(t)$ satisfies. *I can think of two different initial value problems. Don't worry about which one you choose.Don't forget the IC's.*

$y'' + (2\pi)^2 y = \frac{1}{(2\pi)^2 - (4\pi)^2} \cos(4\pi t)$

$y(0) = 0 \qquad \dot{y}(0) = 0$

5 (15 points)

(a) Compute the steady state solution to

$$mu'' + u' + u = \cos t$$

$$u = A\sin t + B\cos t$$

$$\dot{u} = -B\sin t + A\cos t$$

$$+ \quad m\ddot{u} = -Am\sin t - Bm\cos t$$

$$\cos t = [(1-m)A - B]\sin t + [(1-m)B + A]\cos t$$

$$(1-m)A - B = 0 \Rightarrow (1-m)A = B$$

$$(1-m)B + A = 1 \Rightarrow (1-m)^2 A + A = 1$$

$$A = \frac{1}{1+(1-m)^2} \qquad B = \frac{(1-m)}{1+(1-m)^2}$$

$$y(t) = \frac{1}{1+(1-m)^2}\sin t + \frac{(1-m)}{1+(1-m)^2}\cos t$$

(b) Write the steady state solution in amplitude-phase form.

$$y(t) = \frac{1}{\sqrt{1+(1-m)^2}}\cos\left(t - \operatorname{atan}\frac{1}{(1-m)}\right)$$

(c) What value of m yields the maximimu amplitude?

$$m = 1$$

$\boxed{6}$ (20 points) A 1 kg bullet is fired directly upward from ground level ($y = 0$) with velocity $y'(0) = 100$ meters per second. The only forces which act are gravity $-mg$ (set $g = 10$ m/sec^2 to make the numbers easier) and air resistance $-\gamma y'(t)$.

(a) Use Newton's second law to **write a second order differential equation** for the height $y(t)$. Add initial conditions. To make the calculations below easier, change the notation by replacing γ with $\frac{1}{\tau}$. *Hint: If it helps, think of this as a forced damped mass-spring system without the spring (i.e. $k = 0$).*

$$\ddot{y} + \frac{1}{\tau}\dot{y} = -10 \qquad y(0) = 0 \qquad \dot{y}(0) = 100$$

(b) Compute the solution to the initial value problem as a function of t and τ.

$$\ddot{y}_H + \frac{1}{\tau}\dot{y} = 0 \qquad r^2 + \frac{r}{\tau} = 0 \qquad r = 0,\ r = -\frac{1}{\tau}$$

$$y_H = C_1 + C_2 e^{-t/\tau}$$

$$\ddot{y}_p + \frac{1}{\tau}\dot{y}_p = -10$$

Seek $y_p = A$

Homogeneous Question - Yes

$y_p = At$ Homog. Quest. - No

$$\frac{1}{\tau}\dot{y}_p = \frac{1}{\tau}A$$
$$+\ \ddot{y}_p = 0$$
$$-10 = \frac{1}{\tau}A$$
$$A = -10\tau$$

$$y(t) = C_1 + C_2 e^{-t/\tau} - 10\tau t$$
$$0 = C_1 + C_2$$
$$100 = -\frac{C_2}{\tau} - 10\tau$$

$$100\tau + 10\tau^2 = -C_2$$

$$C_2 = -10\tau^2 - 100\tau$$

$$C_1 = 10\tau^2 + 100\tau$$

(continued on next page)

(Problem 6 continued)

$$y(t) = (10\tau^2 + 100\tau)(1 - e^{-t/\tau}) - 10\tau t$$

(c) The bullet lands 15 seconds later. Write down an equation for the coefficient τ.

$$0 = (10\tau^2 + 100\tau)(1 - e^{-15/\tau}) - 10 \cdot 15\tau$$

(d) Make the assumption that $\frac{15}{\tau}$ is big enough that you can ignore the term $e^{\frac{-15}{\tau}}$. Now solve for τ.

$$0 = 10\tau^2 - 50\tau \Rightarrow 0 = \tau(\tau - 5) \Rightarrow \boxed{\tau = 5}$$

Table of Laplace Transforms

1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$u_a(t)$	$\frac{e^{-as}}{s}$
$\delta_a(t)$	e^{-as}
$y'(t)$	$sY(s) - y(0)$
$y''(t)$	$s^2Y(s) - y(0)s - y'(0)$
$e^{at}y(t)$	$Y(s-a)$
$ty(t)$	$-\frac{d}{ds}Y(s)$
$u_a(t)y(t-a)$	$e^{-as}Y(s)$
$u_a(t)y(t)$	$e^{-as}\mathcal{L}\{y(t+a)\}$
$y(at)$	$\frac{1}{a}Y(\frac{s}{a})$

Trig Formulas

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$
$$\cos\theta\cos\phi = \frac{1}{2}\left(\cos(\theta+\phi) + \cos(\theta-\phi)\right)$$
$$\sin\theta\sin\phi = \frac{1}{2}\left(\cos(\theta-\phi) - \cos(\theta+\phi)\right)$$
$$\sin\theta\cos\phi = \frac{1}{2}\left(\sin(\theta+\phi) + \sin(\theta-\phi)\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$
$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$
$$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$$
$$\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$$