Name: ______________________________ ID #: ____________________

Mathematics 307N Final Exam

12 March 2018

Instructions: This is a closed book exam, no notes (other than what I have provided) or calculators allowed. Please turn off all cell phones and similar devices. It's a good idea to put a box around each solution. When solving differential equations, real solutions are preferable to complex ones.

1. (10 points) Compute the Laplace transform of

$$g(t) = \begin{cases} \cos 3t & \text{if } 0 \le t < 2, \\ 4 - t & \text{if } 2 \le t < 6, \\ -2 & \text{if } 6 \le t. \end{cases}$$

2. (10 points) Solve

$$y'' + 6y' + 13y = \delta(t - \pi), \quad y(0) = 0, \; y'(0) = 0.$$

3. (10 points) (a) Compute the inverse Laplace transform of $\frac{s}{s^2 - 4s + 5}$.

(b) Compute the inverse Laplace transform of

$$\frac{6 - 3e^{-3s}}{(s+1)(s+4)}.$$

4. (10 points) Solve
$$y'' - y' = 10\sin 2t, \quad y(0) = 5,\ y'(0) = 0.$$
(You may use either Laplace transforms or the characteristic equation plus undetermined coefficients. Check your work!)

5. (5 points) State Euler's formula (the one about complex numbers).

6. (5 points) Consider the differential equation

$$y'' - 3y' + 2y = (3t^2 - 4t)\cos 6t + 7e^t.$$

DO NOT SOLVE. According to the method of undetermined coefficients, what should you try for the particular solution? (Do not solve for the coefficients, just tell me the form for the solution. You should probably determine y_h first, of course. Check your work for y_h!)

7. (5 points) For the equation

$$y' = (y - 10)(y + 3)^2(y - 2),$$

determine the equilibrium solutions and classify each one as asymptotically stable, unstable or semistable.

8. (5 points) If we have an underdamped mass-spring system described by the equation

$$y'' + \frac{1}{10000}y' + 4y = \cos \omega t$$

(so $m = 1$, $\gamma = 1/10000$, and $k = 4$), then we can write the steady state solution in "amplitude-phase" form, $y = R\cos(\omega t - \delta)$.

If the driving frequency ω is chosen so that the system achieves resonance, what (approximately) is the phase angle?

Trig identities:

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$c_1 \cos \mu t + c_2 \sin \mu t = R \cos(\mu t - \delta),$$
$$\text{where } R = \sqrt{c_1^2 + c_2^2},\ \cos\delta = c_1/R,\ \sin\delta = c_2/R.$$
$$\cos \omega t - \cos \omega_0 t = 2 \sin\left(\frac{\omega_0 - \omega}{2} t\right) \sin\left(\frac{\omega_0 + \omega}{2} t\right).$$

Mass-spring systems: as long as $\omega \neq \omega_0$, a particular solution of $mu'' + ku = F_0 \cos \omega t$ is

$$u_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t.$$

A particular solution of $mu'' + \gamma u' + ku = F_0 \cos \omega t$ is

$$u_p = R \cos(\omega t - \delta),$$
$$\text{where } R = \frac{F_0}{\Delta},\ \cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta},\ \sin\delta = \frac{\gamma\omega}{\Delta},$$
$$\text{and } \Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}.$$

(**Laplace transforms** on the other side.)

Some Laplace transforms

Definition of Laplace transform: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
0	0
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
(where $\Gamma(p+1) = \int_0^\infty e^{-x} x^p dx$)	
$\sin(at)$	$\frac{a}{s^2+a^2}, \quad s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}, \quad s > 0$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$t^n e^{at}$, $n = 1, 2, 3, \ldots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$u_c(t)f(t)$	$e^{-cs}\mathcal{L}\{f(t+c)\}$
$e^{ct}f(t)$	$F(s-c)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
$\delta(t-c)$	e^{-cs}
$\int_0^t f(t-u)g(u)du$	$F(s)G(s)$