MATH 307 J
FINAL EXAM
SPRING 2018

Name ______________________________ Student ID #______________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:______________________________

- Your exam should consist of this cover sheet, followed by 8 problems, a bonus problem, and the table of Laplace transforms. Check that you have a complete exam.
- Pace yourself. You have 1 hour and 50 minutes to complete the exam and there are 8 problems. Try not to spend more than 12 minutes on each page.
- Unless otherwise indicated, show all your work and justify your answers.
- Unless otherwise indicated, your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to 0.7854.)
- You may use a scientific calculator and one doubled-sided 8.5×11-inch sheet of handwritten notes. All other electronic devices (including graphing and programmable calculators and calculators with calculus functions) are forbidden.
- You are not allowed to use your phone, headphones, or earbuds for any reason during this exam. Turn your phone off and put it away for the duration of the exam.
- There may be multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Raise your hand if you have any questions and check your work if you have time left.
- The bonus question may not be graded on the same scale as the exam and should not be thought of as a way to replace a problem from the exam which you don't know how to do. Only spend time on it once you are satisfied with your answers to the rest of the problems.

1	14	
2	12	
3	14	
4	10	
5	16	
6	10	
7	12	
8	12	
Total	100	
Bonus	8	

GOOD LUCK!

1. (a) i. (4 points) Find **two** different solutions to $\cos^2(4t)\dfrac{dy}{dt} = (y-1)^{2/3}$ with $y(0) = 1$.

not covered in all sections

ii. (2 points) Explain why the uniqueness theorem for nonlinear equations does not apply to the above IVP (i.e. show me which hypothesis is not satisfied).

not covered in all sections

(b) (8 points) Use the substitution $v = y^{-1}$ to solve $x\dfrac{dy}{dx} - (1-x)y = y^2$ for $x > 0$.

2. (12 points) A tank with capacity 300 liters initially contains 60 liters of a salt water solution with concentration 0.25 kilograms of salt per liter. Water containing 0.5 kilograms of salt per liter is pumped into the tank at rate of 6 liters per minute. Meanwhile, the mixture in the tank flows out of the tank at a rate of 3 liters per minute.

   (a) Find the amount of salt $A(t)$ in the tank for any time prior to the instant when the tank begins to overflow.

   (b) Find the concentration of salt in the tank when it is on the point of overflowing.

3. (a) i. (5 points) Consider the autonomous equation $\frac{dy}{dt} = (y^3 - 4y)(4 - e^{-y})(y - 2)$. Find and classify all equilibrium solutions as stable, unstable, or semistable.

ii. (3 points) For which value(s) $y(0) = y_0$ does the solution satisfy $\lim_{t \to \infty} y(t) = -2$?

(b) (6 points) Consider the spring-mass system

$$mu'' + ku = F_0 \cos(\omega t)$$

where $F_0 \neq 0$ is constant. Describe the main way that resulting motion differs when $\omega \neq \omega_0$ in comparison to when $\omega = \omega_0$. Here ω_0 denotes the natural frequency of the system. (No need to write an essay; a couple lines should suffice).

4. (10 points) An object weighing 4 lb is attached to a spring that has spring-constant 2 lb/ft. The entire system is submerged in a liquid that imparts a damping force which previous experiments have shown to have a magnitude of 0.625 lb when the object moves at a speed of 0.5 ft/s.

   Now the object is lifted up 3 inches above equilibrium position and released with an initial velocity v.

   (a) (7 points) In terms of v, find the equation of motion of the object.

   (b) (3 points) If the object passes through the equilibrium position after 0.5 seconds, what was the initial velocity of the object? Round your answer to four decimal places.

5. (a) (8 points) For each equation, write down the best guess for the form of the particular solution y_p. Use capital letters $A, B, \dots$ etc as needed for the undetermined coefficients. You DO NOT need to find the values of $A, B, \dots$ etc.

   (i) (4 points) $$y'' + 3y = e^{5t}\sin(3t) + 7 - t^2$$

   (ii) (4 points) $$y'' - 4y' + 4y = 3te^{2t} - 4e^{-2t}$$

   (b) (8 points) Find a differential equation that has

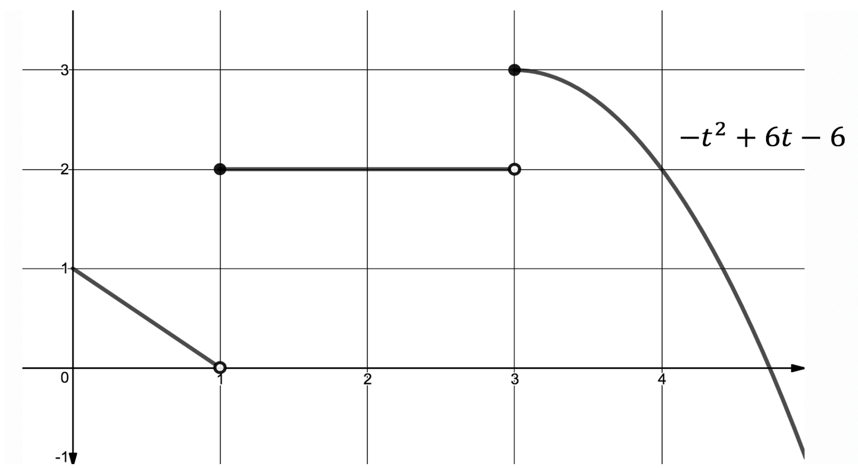
$$y = c_1 e^{-3t}\cos(\sqrt{5}t) + c_2 e^{-3t}\sin(\sqrt{5}t) - e^{-2t} + \frac{1}{2}t^2 - t$$

as its general solution.

6. (a) (4 points) Use the **definition** of the Laplace transform to show for $c > 0$ that

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s).$$

(b) (6 points) Consider the piecewise continuous graph of $f(t)$ on $(0, \infty)$.



Compute the Laplace transform of $f(t)$.

7. (a) (6 points) Use the table of Laplace transforms to find $\mathcal{L}\{te^{at}\sin(bt)\}$. Simplify your answer.

(b) (6 points) Find the inverse Laplace transform $\mathcal{L}^{-1}\left\{\dfrac{3e^{-2s}}{s^5} - \dfrac{s^2 - 2se^{-s}}{s^2 - 3}\right\}$.

8. (12 points) Use the Laplace transform to solve the initial value problem

$$y'' - 6y' = \begin{cases} 4 & \text{if } 0 \le t < 3, \\ 0 & \text{if } 3 \le t < \infty \end{cases}, \qquad y(0) = 2,\ y'(0) = -3$$

Bonus (8 points)

(a) Show that

$$\mathcal{L}\left\{ \int_0^t f(x)\, dx \right\} = \frac{1}{s} F(s).$$

(b) Find a function that satisfies the integral equation

$$\int_0^t f(x)\, dx = 2f(t) + t - 1.$$

Table of Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$\mathcal{L}\{f(t)\} = F(s)$	
1	$\dfrac{1}{s}$	$s > 0$
e^{at}	$\dfrac{1}{s-a}$	$s > a$
$t^n \quad n = 1, 2, 3, \ldots$	$\dfrac{n!}{s^{n+1}}$	$s > 0$
$\sin at$	$\dfrac{a}{s^2 + a^2}$	$s > 0$
$\cos at$	$\dfrac{s}{s^2 + a^2}$	$s > 0$
$\sinh at$	$\dfrac{a}{s^2 - a^2}$	$s > \lvert a \rvert$
$\cosh at$	$\dfrac{s}{s^2 - a^2}$	$s > \lvert a \rvert$
$e^{at} \sin bt$	$\dfrac{b}{(s-a)^2 + b^2}$	$s > a$
$e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}$	$s > a$
$t^n e^{at} \quad n = 1, 2, 3, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}$	$s > a$
$u_c(t)$	$\dfrac{e^{-cs}}{s}$	$s > 0$
$u_c(t) f(t-c)$	$e^{-cs} F(s)$	
$e^{ct} f(t)$	$F(s-c)$	
$f(ct)$	$\dfrac{1}{c} F\left(\dfrac{s}{c}\right)$	$c > 0$
$(f*g)(t) = \int_0^t f(t-\tau) g(\tau)\, d\tau$	$F(s)G(s)$	
$\delta(t-c)$	e^{-cs}	
$f^{(n)}(t) \quad n = 1, 2, 3, \ldots$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$	
$t^n f(t) \quad n = 1, 2, 3, \ldots$	$(-1)^n F^{(n)}(s)$	