Spring 2018 MATH 307 Final Exam
90 pts total

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Instruction:

- Nothing but writing utensils and a double side 4in $\times$ 6in notecard are allowed.
- Use the provided Table of Laplace Transforms.
- Unless otherwise specified, you must show work to receive full credit.

1. (13pts) Consider the initial value problem

$$\frac{dy}{dx} = \frac{2x+1}{y-1}, \qquad y(0) = -1.$$

Find $y(1)$.

$$(y-1)\,dy = (2x+1)\,dx$$

$$\frac{y^2}{2} - y = x^2 + x + C$$

$$y^2 - 2y = 2x^2 + 2x + B$$

$$y(0) = -1 \Rightarrow 3 = C$$

$$y(x) = \frac{2 \pm \sqrt{4 + 4(2x^2+2x+3)}}{2}$$

$$y(x) = 1 \pm \sqrt{4 + 2x^2 + 2x}$$

$$y(0) = -1 \Rightarrow y(x) = 1 - \sqrt{4 + 2x^2 + 2x}$$

$$y(1) = 1 - 2\sqrt{2}$$

2. (15pts) Newton's law of cooling states that the rate of change of temperature of an object in a surrounding medium is proportional to the difference of the temperature of the medium and the temperature of the object.

Suppose a metal bar, initially at temperature $T(0) = 40$ degrees Celsius, is placed in a room which is held at the constant temperature of 20 degrees Celsius. One minute later the bar has cooled to 30 degrees. Find $T(t)$ for all time $t > 0$.

Hint: first write the differential equation that models the temperature (in degrees Celsius) as a function of time (in minutes). Start by calling the constant of proportionality k. Solving the initial value problem to obtain the temperature as a function of k and t. Then use the observed temperature after one minute to solve for k.

$$\frac{dT}{dt} = -k\,(T - 20)$$

doesn't matter if you didn't put a "−" sign here in the beginning. You'll just end up finding a negative value for k later on.

$$\frac{dT}{T-20} = -k\,dt$$

$$\ln|T-20| = -kt + C$$

$$T - 20 = Ae^{-kt}$$

$$T = 20 + Ae^{-kt}$$

$$T(0) = 40 \Rightarrow 40 = 20 + A \Rightarrow A = 20$$

$$T(t) = 20 + 20e^{-kt}$$

$$T(1) = 30 = 20 + 20e^{-k}$$

$$e^{-k} = \frac{10}{20} = \frac{1}{2}$$

$$-k = \ln\left(\tfrac{1}{2}\right) = -\ln 2$$

(or can write
$$T(t) = 20 + 20e^{-kt} = 20 + 20(e^{-k})^t = 20 + 20\left(\tfrac{1}{2}\right)^t$$
)

$$T(t) = 20 + 20e^{-(\ln 2)t}$$

3. (13pts) Consider the differential equation

$$y'' - 2y' + 2y = te^t \cos t + e^t + \sin t + 1.$$

Write down the form of a particular solution (i.e. the Ansatz), no need to find the constants.

$$r^2 - 2r + 2 = 0$$

$$r = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

Ansatz:

$$y_p(t) = t\Big((At+B)e^t \cos t + (Ct+D)e^t \sin t\Big) + Ee^t + F\cos t + G\sin t + H$$

4. (15pts) You have a spring in a damping media, but you don't know the spring constant nor the damping constant. To find that out, you decide to attach a mass of 1 kg to the spring and plot the motion of this unforced damped spring-mass system. The graph below is a plot of the displacement of the mass at any time t. Write down the differential equation governing its motion.

Note: you should write down actual (estimated) numbers based on what you gather from the graph, not just a symbolic equation.

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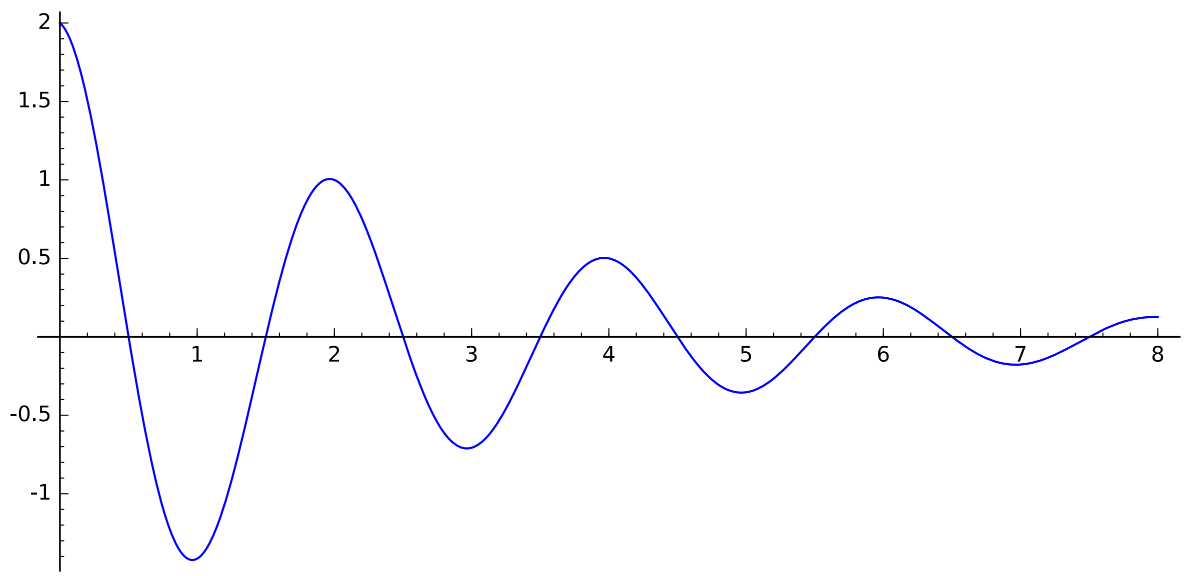


free vibration w/ damping:

$m = 1$

$$y'' + \gamma y' + ky = 0$$

char eq: $-\frac{\gamma}{2} \pm \frac{\sqrt{\gamma^2 - 4k}}{2} = -\frac{\gamma}{2} \pm i \underbrace{\frac{\sqrt{4k-\gamma^2}}{2}}_{=\mu}$

Soln: $y(t) = C_1 e^{-\frac{\gamma}{2}t} \cos \mu t + C_2 e^{-\frac{\gamma}{2}} \sin \mu t$

$$= A e^{-\frac{\gamma}{2}t} \cos(\mu t - \delta)$$

Can immediately read off from the picture that

$\boxed{A=2, \quad \delta = 0,}$ (quasi)-period $= 2 \Rightarrow \mu = \pi$

* see comment below

$\Rightarrow$ continue next page

(Extra space in case you need it.)

So, $y(t) = 2e^{-\frac{\gamma}{2}t}\cos(\pi t)$

To figure out γ:

$$\underbrace{1 = y(2)}_{\text{read off the graph}} = 2e^{-\frac{\gamma}{2}(2)}\cos(2\pi) = 2e^{-\gamma}$$

$$e^{-\gamma} = \frac{1}{2}$$

$$-\gamma = \ln\left(\frac{1}{2}\right) = -\ln(2)$$

$$\gamma = \ln(2) \approx 0.69$$

To figure out k:

$$\pi = \mu = \frac{\sqrt{4k-\gamma^2}}{2}$$

$$\Rightarrow 4\pi^2 = 4k - (\ln(2))^2$$

$$\Rightarrow \quad k = \pi^2 + \left(\frac{\ln(2)}{2}\right)^2 \approx 9.99$$

$$\boxed{y'' + \ln(2)\, y' + \left(\pi^2 + \left(\frac{\ln(2)}{2}\right)^2\right) y = 0}$$

* The approximation delta = 0 is certainly reasonable based on the graph, but it might occur to you that the function Aexp(-(gamma/2)t) cos(pi t) doesn't have a maximum at t=0, so delta can't be exactly zero and A can't be exactly 2. Notice that the calculation of gamma doesn't require knowledge of the specific values of A and delta.

$$A\cos\delta = A\cos(\pi\cdot 0 - \delta) = y(0) = 2$$

$$(A\cos\delta)e^{-\gamma} = A\cos(\pi\cdot 2 - \delta)e^{-\frac{\gamma}{2}2} = y(2) = 1$$

Now dividing gives $e^{-\gamma} = 2$ or $\gamma = \ln(2)$

5. (12pts) Find the Laplace transform

$$\mathcal{L}\{2y''(t) - ty'(t) + 3y(t)\},$$

in terms of $Y(s) = \mathcal{L}\{y\}$ and $Y'(s)$, where $y(0) = 2$ and $y'(0) = -1$.
Hint: use #16 in the table of Laplace transforms.

$$\mathcal{L}\{2y''(t) - ty'(t) + 3y(t)\}$$

$$= 2\mathcal{L}\{y''\} + \underline{\mathcal{L}\{-ty'\}} + 3\mathcal{L}\{y\}$$

$$= 2\left(s^2Y(s) - sy(0) - y'(0)\right) + \underbrace{\frac{d}{ds}\mathcal{L}\{y'\}}_{\text{using \#16}} + 3Y(s)$$

$$= 2\left(s^2Y(s) - 2s + 1\right) + \frac{d}{ds}\left(sY(s) - \underbrace{y(0)}_{=2}\right) + 3Y(s)$$

$$= 2s^2Y(s) - 4s + 2 + \left(sY'(s) + Y(s)\right) + 3Y(s)$$

$$= sY'(s) + (2s^2 + 4)Y(s) - 4s + 2$$

6. Consider the mechanical vibration modeled by the initial value problem:

$$y''(t) + \pi^2 y(t) = \pi\delta(t - 1.5), \qquad y(0) = -1, \; y'(0) = 0.$$

[a] **(14pts)** Find the solution to this initial value problem.
Hint: use Laplace transform.

$$s^2 Y - s\underbrace{y(0)}_{=-1} - \underbrace{y'(0)}_{=0} + \pi^2 Y(s) = \pi e^{-1.5s}$$

$$(s^2 + \pi^2) Y(s) + s = \pi e^{-1.5s}$$

$$Y(s) = \frac{\pi e^{-1.5s}}{s^2 + \pi^2} - \frac{s}{s^2+\pi^2}$$

$$Y(s) = e^{-1.5s} \underbrace{\frac{\pi}{s^2+\pi^2}}_{= H(s)} - \frac{s}{s^2+\pi^2}$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \sin \pi t$$

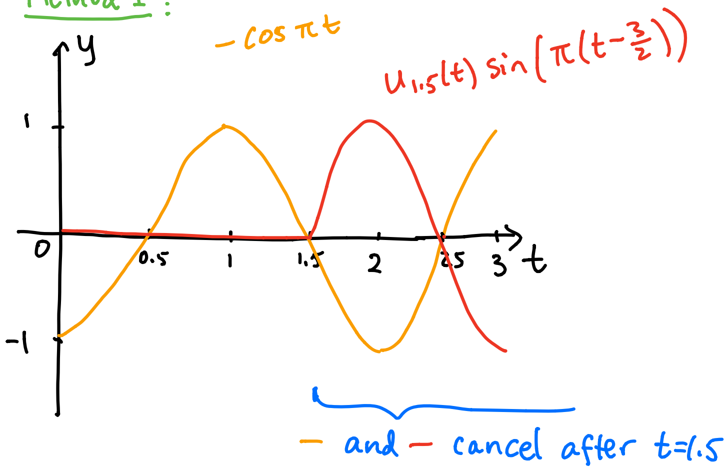
$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = u_{1.5}(t)\, h(t-1.5) - \cos \pi t$$

$$\boxed{y(t) = u_{1.5}(t) \sin\left(\pi\left(t - \tfrac{3}{2}\right)\right) - \cos(\pi t)}$$

$$= \begin{cases} -\cos(\pi t), & 0 \le t < 1.5 \\ \sin\left(\pi\left(t - \tfrac{3}{2}\right)\right) - \cos(\pi t), & t \ge 1.5 \end{cases}$$

For part (b):

Method 1:



Method 2

$$\sin\left(x - \tfrac{3\pi}{2}\right) = \cos(x)$$

So $y(t) = \begin{cases} -\cos(\pi t), & 0 \le t < 1.5 \\ \underbrace{\cos(\pi t) - \cos(\pi t)}_{=0}, & t \ge 1.5 \end{cases}$

6 (continue)

[b] (8pts) Graph the solution found in part [a].

Hint: if you couldn't solve part [a], you can still attempt to graph the solution as much as possible to earn partial credits.