Name: Solutions ID #: $e^{i\pi/4}$

# Mathematics 307N Final Exam

12 March 2018

**Instructions**: This is a closed book exam, no notes (other than what I have provided) or calculators allowed. Please turn off all cell phones and similar devices. It's a good idea to put a box around each solution. When solving differential equations, real solutions are preferable to complex ones.

1. (10 points) Compute the Laplace transform of

$$g(t) = \begin{cases} \cos 3t & \text{if } 0 \le t < 2, \\ 4 - t & \text{if } 2 \le t < 6, \\ -2 & \text{if } 6 \le t. \end{cases}$$

**Solution:** First write $g(t)$ in terms of step functions:

$$\begin{aligned} g(t) &= \cos 3t + u_2(t)(-\cos 3t + 4 - t) + u_6(t)(-4 + t - 2) \\ &= \cos 3t + u_2(t)(-\cos 3t + 4 - t) + u_6(t)(t - 6). \end{aligned}$$

Now apply the Laplace transform, using the entry on the table $u_c(t)f(t) \leftrightarrow e^{-cs}\mathcal{L}\{f(t+c)\}$. For the first step function, the function "$f(t)$" is $f(t) = -\cos 3t + 4 - t$, and we need to take the Laplace transform of

$$f(t+2) = -\cos 3(t+2) + 4 - (t+2) = -\cos 6 \cos 3t + \sin 6 \sin 3t + 2 - t.$$

For the second step function, the function "$f(t)$" is $f(t) = t - 6$, and we need to take the Laplace transform of $f(t+6) = t$. So putting all of this together, we get

$$\boxed{G(s) = \frac{s}{s^2+9} + e^{-2s}\left(-\cos 6 \frac{s}{s^2+9} + \sin 6 \frac{3}{s^2+9} + \frac{2}{s} - \frac{1}{s^2}\right) + e^{-6s}\left(\frac{1}{s^2}\right)}.$$

2. (10 points) Solve

$$y'' + 6y' + 13y = \delta(t - \pi), \quad y(0) = 0,\ y'(0) = 0.$$

**Solution:** Apply the Laplace transform to the equation, as usual writing $Y$ for the Laplace transform of $y$:

$$(s^2 + 6s + 13)Y = e^{-\pi s},$$

so

$$Y = e^{-\pi s}\frac{1}{s^2 + 6s + 13} = e^{-\pi s}\frac{1}{(s+3)^2 + 2^2}.$$

The inverse Laplace transform of the fraction $F(s)$ is $f(t) = \frac{1}{2}e^{-3t}\sin 2t$. Since $F(s)$ is multiplied by an exponential function, the inverse Laplace transform of the product is

$$\begin{aligned} \boxed{y} &= u_\pi(t)f(t-\pi) = \boxed{u_\pi(t)\frac{1}{2}e^{-3(t-\pi)}\sin 2(t-\pi)} \\ &= \boxed{u_\pi(t)\frac{1}{2}e^{3\pi - 3t}\sin 2t} \end{aligned}$$

(since $\sin(2t - 2\pi) = \sin 2t$).

3. (10 points) (a) Compute the inverse Laplace transform of $\dfrac{s}{s^2-4s+5}$.

**Solution:** Complete the square in the denominator:

$$\frac{s}{s^2-4s+4} = \frac{s}{(s-2)^2+1^2} = \frac{s-2+2}{(s-2)^2+1^2} = \frac{s-2}{(s-2)^2+1^2} + \frac{2}{(s-2)^2+1^2}.$$

Now we can read off the inverse Laplace transform from the table:

$$\boxed{e^{2t}\cos t + 2e^{2t}\sin t}.$$

(b) Compute the inverse Laplace transform of

$$\frac{6-3e^{-3s}}{(s+1)(s+4)}.$$

**Solution:** Do some simple algebra and then use partial fractions:

$$\begin{aligned}
\frac{6-3e^{-3s}}{(s+1)(s+4)} &= (2-e^{-3s})\frac{3}{(s+1)(s+4)} = (2-e^{-3s})\left(\frac{A}{s+1}+\frac{B}{s+4}\right) \\
&= (2-e^{-3s})\left(\frac{1}{s+1}+\frac{-1}{s+4}\right) \\
&= 2\left(\frac{1}{s+1}+\frac{-1}{s+4}\right) - e^{-3s}\left(\frac{1}{s+1}+\frac{-1}{s+4}\right).
\end{aligned}$$

The first term has inverse Laplace transform $2f(t) = 2(e^{-t}-e^{-4t})$, and the second has inverse Laplace transform $u_3(t)f(t-3)$ (with the same function $f$). Putting these together yields the answer:

$$\boxed{2(e^{-t}-e^{-4t}) - u_3(t)(e^{-(t-3)}-e^{-4(t-3)})},$$

or, if you prefer,

$$\boxed{2(e^{-t}-e^{-4t}) - u_3(t)(e^{3-t}-e^{12-4t})},$$

or, if you prefer,

$$\boxed{\begin{cases} 2(e^{-t}-e^{-4t}) & \text{if } t<3, \\ 2(e^{-t}-e^{-4t}) - (e^{3-t}-e^{12-4t}) & \text{if } t \geq 3. \end{cases}}$$

4. (10 points) Solve
$$y'' - y' = 10\sin 2t, \quad y(0) = 5,\ y'(0) = 0.$$
(You may use either Laplace transforms or the characteristic equation plus undetermined coefficients. Check your work!)

**Solution:** Using the Laplace transform:
$$\begin{aligned}
(s^2Y - sy(0) - y'(0)) - (sY - y(0)) &= \frac{20}{s^2+4} \\
(s^2 - s)Y - 5s + 5 &= \frac{20}{s^2+4} \\
(s^2 - s)Y &= 5s - 5 + \frac{20}{s^2+4} \\
Y &= \frac{5s-5}{s^2-s} + \frac{20}{(s^2+4)(s^2-s)} \\
Y &= \frac{5s-5}{s(s-1)} + \frac{20}{(s^2+4)s(s-1)} \\
Y &= \frac{(5s-5)(s^2+4)+20}{(s^2+4)s(s-1)}
\end{aligned}$$
Now use partial fractions:
$$\begin{aligned}
Y &= \frac{5s^3 - 5s^2 + 20s}{(s^2+4)s(s-1)} \\
&= \frac{As+B}{s^2+4} + \frac{C}{s} + \frac{D}{s-1} \\
&= \frac{s-4}{s^2+4} + \frac{0}{s} + \frac{4}{s-1} \\
&= \frac{s}{s^2+4} - 2\frac{2}{s^2+4} + \frac{0}{s} + \frac{4}{s-1}
\end{aligned}$$
So the solution is
$$\boxed{y = \cos 2t - 2\sin 2t + 4e^t}.$$

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Using the characteristic equation and undetermined coefficients: the characteristic equation is $r^2 - r = 0$, which has roots 0 and 1, so the solution to the associated homogeneous equation is
$$y_h = c_1 e^t + c_2.$$
For the particular solution, try
$$y_p = A\cos 2t + B\sin 2t.$$

Since no summands of this occur in $y_h$, we're good to go. So plug in and solve for $A$ and $B$:

$$y_p' = -2A\sin 2t + 2B\cos 2t,$$
$$y_p'' = -4A\cos 2t - 4B\sin 2t,$$

so the differential equation becomes

$$(-4A - 2B)\cos 2t + (-4B + 2A)\sin 2t = 10\sin 2t.$$

Therefore $-4A - 2B = 0$ and $-4B + 2A = 10$, so use algebra to deduced $A = 1$ and $B = -2$. So $y_p = \cos 2t - 2\sin 2t$, and the general solution is

$$y = c_1 e^t + c_2 + \cos 2t - 2\sin 2t.$$

Now use the initial conditions to find the constants. Note that $y' = c_1 e^t - 2\sin 2t - 4\cos 2t$, so the initial conditions $y(0) = 5$ and $y'(0) = 0$ become

$$c_1 + c_2 + 1 = 5 \quad c_1 - 4 = 0.$$

So $c_4 = 4$ and $c_2 = 0$, and the solution is

$$\boxed{y = 4e^t + \cos 2t - 2\sin 2t}.$$

5. (5 points) State Euler's formula (the one about complex numbers).

**Solution:** $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

6. (5 points) Consider the differential equation

$$y'' - 3y' + 2y = (3t^2 - 4t)\cos 6t + 7e^t.$$

DO NOT SOLVE. According to the method of undetermined coefficients, what should you try for the particular solution? (Do not solve for the coefficients, just tell me the form for the solution. You should probably determine $y_h$ first, of course. Check your work for $y_h$!)

**Solution:** The characteristic equation is $r^2 - 3r + 2 = 0$, which has roots 1 and 2, so $y_h = c_1e^t + c_2e^{2t}$. Our first guess for the particular solution should be

$$y_p = (At^2 + Bt + C)\cos 6t + (Dt^2 + Et + F)\sin 6t + Ge^t,$$

but the last summand is part of $y_h$, so it needs to be multiplied by $t$. So the correct form is

$$\boxed{y_p = (At^2 + Bt + C)\cos 6t + (Dt^2 + Et + F)\sin 6t + Gte^t}.$$

Note that $Y_{bad} = (At^2 + Bt + C)(D\cos 6t + E\sin 6t)$ will not work for the first part: you need 6 independent coefficients for the terms $t^2\cos 6t$, $t\cos 6t$, etc., and $Y_{bad}$ does not give 6 independent coefficients (since it only uses 5 coefficients).

7. (5 points) For the equation

$$y' = (y-10)(y+3)^2(y-2),$$

determine the equilibrium solutions and classify each one as asymptotically stable, unstable or semistable.

**Solution:** The equilibrium solutions are $y = 10$, $y = -3$ and $y = 2$. Then we can look at cases:

- if $y < -3$, then $y' > 0$,
- if $-3 < y < 2$, then $y' > 0$,
- if $2 < y < 10$, then $y' < 0$, and
- if $10 < y$, then $y' > 0$.

So:

- $y = -3$ is a semistable equilibrium.
- $y = 2$ is a stable equilibrium.
- $y = 10$ is an unstable equilibrium.

8. (5 points) If we have an underdamped mass-spring system described by the equation

$$y'' + \frac{1}{10000}y' + 4y = \cos \omega t$$

(so $m = 1$, $\gamma = 1/10000$, and $k = 4$), then we can write the steady state solution in "amplitude-phase" form, $y = R\cos(\omega t - \delta)$.

If the driving frequency $\omega$ is chosen so that the system achieves resonance, what (approximately) is the phase angle?

**Solution:** Since the system has very little damping, then to achieve resonance, we should choose $\omega$ to be approximately equal to $\omega_0 = 2$. If we consult the formula sheet, we find that the phase angle $\delta$ satisfies

$$\cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\sqrt{m^2(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}},$$
$$\sin\delta = \frac{\gamma\omega}{\sqrt{m^2(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}}$$

If $\omega \approx 2$, then this becomes

$$\cos\delta \approx 0,$$
$$\sin\delta \approx \frac{\gamma\omega}{\sqrt{\gamma^2\omega^2}} \approx 1.$$

Therefore $\boxed{\delta \approx \pi/2}$.

Note that if we only knew that $\cos\delta \approx 0$, we could not tell whether $\delta$ was (approximately) $\pi/2$ or $-\pi/2$ – we need to know that $\sin\delta \approx 1$ to resolve this ambiguity.