Math 307 G - Spring 2018
Final Exam
June 5, 2018

Name: ____________________

Section: ____________________

Student ID Number: ____________________

- There are 11 pages in total. A Laplace Transform Table is on Page 2. Make sure your exam contains all these questions.
- You are allowed to use a scientific calculator (**no graphing calculators and no calculators that have calculus capabilities**) and one **hand-written** 8.5 by 11 inch page of notes.
- You must show your work on all problems. The correct answer with no supporting work may result in no credit. **Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.** Give exact answers wherever possible.
- If you need more room, use the backs of the pages and indicate to the grader that you have done so.
- Raise your hand if you have a question.
- You have 110 minutes to complete the exam. Budget your time wisely.

Problem 1	12	
Problem 2	12	
Problem 3	10	
Problem 4	10	
Problem 5	12	
Problem 6	8	
Problem 7	12	
Problem 8	10	
Problem 9	14	
Total	100	

GOOD LUCK!

Laplace Transform Table

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
t^n	$\frac{n!}{s^{n+1}}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$u_c(t)$	$\frac{e^{-cs}}{s}$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$\delta_c(t)$	e^{-cs}
$(f*g)(t)$	$F(s)G(s)$

- $\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0)$
- $\mathcal{L}\{y''(t)\} = s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0)$
- $(f*g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau$

1. (12 pts)

(a) (6 pts) Find the Laplace transform of $f(t) = (t-2)^2 - u_2(t)\sin(3t)$

$$f(t) = t^2 - 4t + 4 - U_2 \sin(3t)$$

$$\mathcal{L}\{f(t)\} = \frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s} - e^{-2s}\mathcal{L}\{\sin 3(t+2)\}$$

$$= \frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s} - e^{-2s}\mathcal{L}\{\sin 3(t+2)\}$$

$$= \frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s} - e^{-2s}\mathcal{L}\{\sin(3t+6)\}$$

$$= \frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s} - e^{-2s}\mathcal{L}\{\sin(3t)\cos(6) + \sin(6)\cos(3t)\}$$

$$= \frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s} - e^{-2s}\left[\frac{3}{s^2+9}\cdot\cos(6) + \sin(6)\cdot\frac{s}{s^2+9}\right]$$

(b) (6 pts) Find the inverse Laplace transform of $F(s) = \dfrac{e^{-3(s-2)}}{s^2+16}$.

$$F(s) = \frac{e^6}{4}\cdot e^{-3s}\cdot\frac{4}{s^2+16}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{e^6}{4}\cdot U_3(t)\left[\sin(4t)\right](t-3)$$

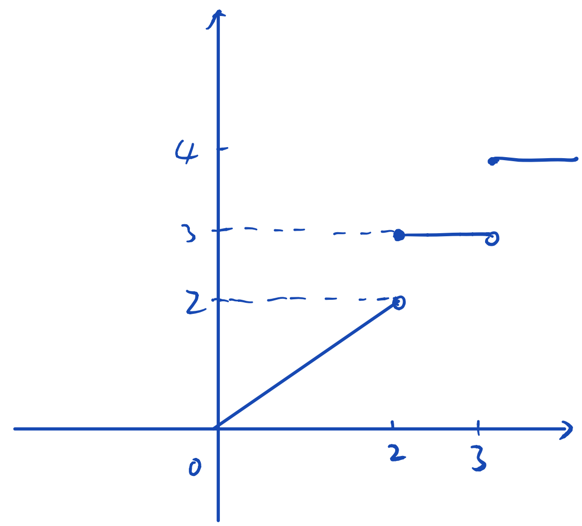
$$= \frac{e^6}{4} U_3(t)\sin(4(t-3))$$

2. (12 pts)

(a) (6 pts) Draw the graph of the function

$$f(t) = t + u_2(t)(3-t) + u_3(t), \quad t \geq 0.$$

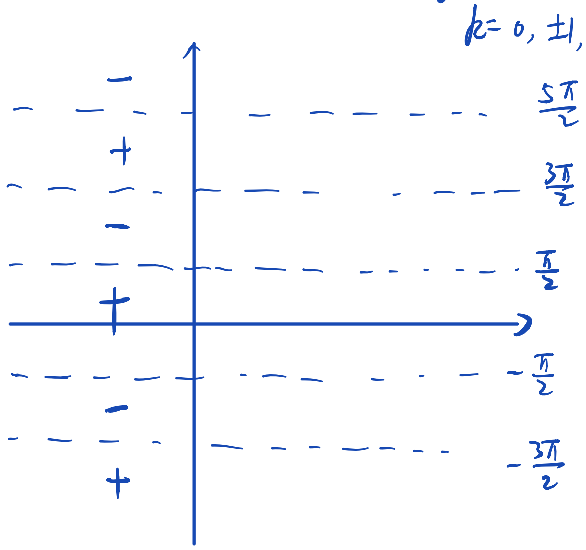
$$f(t) = \begin{cases} t & 0 \leq t < 2 \\ 3 & 2 \leq t < 3 \\ 4 & t \geq 3 \end{cases}$$



(b) (6 pts) Find all the equilibrium solutions of the equation $y' = \cos(y)$ and decide whether they are stable, unstable or semi-stable.

$\cos(y) = 0 \qquad y = k\pi + \frac{\pi}{2}$

$k = 0, \pm 1, \pm 2, \ldots$



stable: $\quad y = 2k\pi + \frac{\pi}{2}$

unstable $\quad y = 2k\pi + \frac{3\pi}{2}$

$k = 0, \pm 1, \pm 2, \pm 3, \ldots$

3. (10 pts) Let $y(t)$ be the solution of the following differential equation

$$y'' + 3y' + 2y = 0, y(0) = 2, y'(0) = -1.$$

(a) (5 pts) Find the solution $y(t)$.

$r^2 + 3r + 2 = 0 \qquad (r+1)(r+2) = 0 \qquad r_1 = -1, \quad r_2 = -2$

$y = c_1 e^{-t} + c_2 e^{-2t}$

$$\begin{cases} y(0) = c_1 + c_2 = 2 \\ y'(0) = -c_1 - 2c_2 = -1 \end{cases} \Rightarrow \begin{cases} c_1 = 3 \\ c_2 = -1 \end{cases}$$

$y = 3e^{-t} - e^{-2t}$

(b) (5 pts) Find the maximum value of $y(t)$.

$y' = -3e^{-t} + 2e^{-2t} = 0 \Rightarrow 2e^{-2t} = 3e^{-t} \Rightarrow e^{t} = \frac{2}{3}$

$t = \ln\left(\frac{2}{3}\right)$

$y(t) = 3e^{-\ln(\frac{2}{3})} - e^{-2\ln(\frac{2}{3})} = 3 \cdot \frac{3}{2} - \left(e^{\ln(\frac{2}{3})}\right)^{-2}$

$= \frac{9}{2} - \left(\frac{2}{3}\right)^{-2} = \frac{9}{2} - \left(\frac{3}{2}\right)^{2} = \frac{9}{2} - \frac{9}{4} = \frac{9}{4}$

max of $y(t) = \frac{9}{4}$.

4. (10 pts)

(a) (5 pts) Derive a formula for $\mathcal{L}\{y'''(t)\}$ in terms of $\mathcal{L}\{y(t)\}, y(0), y'(0)$ and $y''(0)$.

$$\mathcal{L}\{(y''(t))'\} = s\mathcal{L}\{y''(t)\} - y''(0)$$

$$= s(s^2\mathcal{L}\{y\} - sy(0) - y'(0)) - y''(0)$$

$$= s^3\mathcal{L}\{y\} - s^2 y(0) - sy'(0) - y''(0).$$

(b) (5 pts) Use the definition of Laplace transform

$$\mathcal{L}\{y\} = \int_0^\infty e^{-st} f(t)dt$$

to show $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$. You are not allowed to use the conclusion from the Laplace Transform Table for this problem.

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt = \frac{-1}{s-a} e^{-(s-a)t} \Big|_0^\infty$$

$$= \frac{1}{s-a}.$$

5. (12 pts) Use Laplace transforms to find the solution of the initial value problem

$$y'' + 4y = \delta_3(t) + u_3(t)$$

with $y(0) = 0, y'(0) = 0$.

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = e^{-3s} + \frac{e^{-3s}}{s}$$

$$(s^2+4)\mathcal{L}\{y\} = e^{-3s} + \frac{e^{-3s}}{s}$$

$$\mathcal{L}\{y\} = e^{-3s} \cdot \frac{1}{s^2+4} + \frac{e^{-3s}}{s(s^2+4)}$$

$$= \frac{e^{-3s}}{2} \frac{2}{s^2+4} + \frac{e^{-3s}}{4} \cdot \left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$y = U_3(t)\left[\frac{1}{2} \cdot \sin(2t)\right](t-3) + \frac{1}{4} U_3(t)\left[1 - \cos(2t)\right](t-3)$$

$$= U_3(t)\left[\frac{1}{2}\sin(2(t-3))\right] + \frac{1}{4} U_3(t)\left[1 - \cos(2(t-3))\right]$$

6. (8 pts) A spherical raindrop evaporates at a rate proportional to its surface area with (positive) constant of proportionality k; i.e. the rate of change of the volume exactly equals $-k$ times the surface area. **Write a differential equation for the volume V of the drop as a function of time t.** (Hint: The surface area S of a sphere with radius r is $S = 4\pi r^2$. The volume of a spherical raindrop with radius r is $V = \dfrac{4}{3}\pi r^3$.)

$$\frac{dV}{dt} = -kS$$

$$V = \frac{4}{3}\pi r^3 \qquad\qquad S = 4\pi r^2$$

$$\Rightarrow \quad r = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} \qquad\qquad = 4\pi \left(\frac{3V}{4\pi}\right)^{\frac{2}{3}}$$

$$\frac{dV}{dt} = -k \cdot 4\pi \left(\frac{3V}{4\pi}\right)^{\frac{2}{3}}$$

$$= -k\,(4\pi)^{\frac{1}{3}}\,(3V)^{\frac{2}{3}}$$

7. (12 pts) The following are two different problems about spring-mass systems.

(a) (6 pts) Consider a spring-mass system with $m = 2$ kg, $k = 1$ N/m, $\gamma = 1$ N·s/m. Suppose when $0 \le t < 2$, there is no external force. When $2 \le t < 4$, the external force is $2N$. When $t \ge 4$, the external force vanishes. Write down a differential equation (**without solving it**) for the displacement $u(t)$ of the spring from rest at time t. **you answer should include Heaviside step functions** $u_c(t)$.

$$2u'' + u' + u = F(t)$$

$$F(t) = 2U_2(t) - 2U_4(t)$$

$$2u'' + u' + u = 2U_2(t) - 2U_4(t).$$

(b) (6 pts) The displacement $u(t)$ of a damped spring-mass system with constant external force satisfies $u'' + \gamma u' + 4u = 8$, where $\gamma > 0$. Find $\lim_{t\to+\infty} u(t)$.

Since $\gamma > 0$, limit of $u(t)$ only depend on the particular solution of the equation. Let $u = A$

$4A = 8 \qquad A = 2$

$\Rightarrow$ $u(t) = 2$ is a particular solution

$$\lim_{t \to +\infty} u(t) = 2$$

8. (10 pts) Find the implicit solution to the first order differential equation

$$\frac{dy}{dx} - \frac{y}{2x} = e^x y^3, \quad y \neq 0.$$

Hint: Let $u = y^{-2}$.

Let $u = y^{-2}$

$$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx}$$

$$-\frac{1}{2}y^3 \cdot \frac{du}{dx} - \frac{y}{2x} = e^x y^3$$

$$-\frac{1}{2}\frac{du}{dx} - \frac{u}{2x} = e^x$$

$$\frac{du}{dx} + \frac{u}{x} = -2e^x$$

$$\frac{d}{dx}(ux) = \int -2xe^x\,dx$$

$$= -2\int xe^x\,dx$$

$$= -2\left(\int x\,de^x\right)$$

$$= -2(xe^x - e^x) + c$$

$$\Rightarrow \quad ux = -2xe^x - e^x + c$$

$$u = -2e^x - \frac{e^x}{x} + \frac{c}{x}$$

$$y^{-2} = \left(-2 - \frac{1}{x}\right)e^x + \frac{c}{x}.$$

9. (14 pts)

(a) (4 pts) Compute $h(t) = (f * g)(t)$, where $f(t) = t^2, g(t) = t$.

$$(f*g)(t) = \int_0^t \tau^2(t-\tau)\,d\tau = \int_0^t \tau^2 t - \tau^3 d\tau = \frac{1}{3}t^4 - \frac{1}{4}t^4 = \frac{1}{12}t^4.$$

(b) (6 pts) Consider the function $H(s) = \dfrac{1}{s(s^2+1)}$. Find the inverse Laplace transform of $H(s)$ using the convolution.

$$H(s) = \frac{1}{s} \cdot \frac{1}{s^2+1} = F(s)\,G(s)$$

$$f(t) = 1 \qquad g(t) = \sin(t).$$

$$h(s) = (f*g)(t) = \int_0^t 1 \sin(\tau)\,d\tau = -\cos(\tau)\Big|_0^t = 1 - \cos(t).$$

(c) (4 pts) Find a function $f \neq 0$ such that $(f * 1)(t) = f(t) + 1$.

Sol 1

$$f*1(t) = \int_0^t f(\tau)\,d\tau = f(t) + 1 \qquad (*)$$

$$f(t) = f'(t) \qquad f(t) = Ce^t$$

Take $t=0$ in $(*)$

$\Rightarrow$ $0 = f(0) + 1$

$\Rightarrow$ $f(0) = -1$

$\Rightarrow$ $f(t) = -e^t.$

Sol 2:

$$\mathcal{L}\{f\} \cdot \mathcal{L}\{1\} = \mathcal{L}\{f\} + \mathcal{L}\{1\}$$

$$\mathcal{L}\{f\} \cdot \frac{1}{s} = \mathcal{L}\{f\} + \frac{1}{s}$$

$$\left(\frac{1}{s} - 1\right)\mathcal{L}\{f\} = \frac{1}{s}$$

$$\mathcal{L}\{f\} = \frac{\frac{1}{s}}{\frac{1}{s}-1} = \frac{1}{1-s} = -\frac{1}{s-1}$$

$$f = -\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= -e^t.$$