Math 307 G - Spring 2018
Final Exam
June 5, 2018

Name: _____

Section: _____

Student ID Number: _____

- There are 11 pages in total. A Laplace Transform Table is on Page 2. Make sure your exam contains all these questions.
- You are allowed to use a scientific calculator (**no graphing calculators and no calculators that have calculus capabilities**) and one **hand-written** 8.5 by 11 inch page of notes.
- You must show your work on all problems. The correct answer with no supporting work may result in no credit. **Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.** Give exact answers wherever possible.
- If you need more room, use the backs of the pages and indicate to the grader that you have done so.
- Raise your hand if you have a question.
- You have 110 minutes to complete the exam. Budget your time wisely.

Problem 1	12	
Problem 2	12	
Problem 3	10	
Problem 4	10	
Problem 5	12	
Problem 6	8	
Problem 7	12	
Problem 8	10	
Problem 9	14	
Total	100	

GOOD LUCK!

Laplace Transform Table

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
t^n	$\frac{n!}{s^{n+1}}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$u_c(t)$	$\frac{e^{-cs}}{s}$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$\delta_c(t)$	e^{-cs}
$(f*g)(t)$	$F(s)G(s)$

- $\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0)$
- $\mathcal{L}\{y''(t)\} = s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0)$
- $(f*g)(t) = \displaystyle\int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau$
- $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

1. (12 pts)

(a) (6 pts) Find the Laplace transform of $f(t) = (t-2)^2 - u_2(t)\sin(3t)$

(b) (6 pts) Find the inverse Laplace transform of $F(s) = \dfrac{e^{-3(s-2)}}{s^2+16}$.

2. (12 pts)

(a) (6 pts) Draw the graph of the function

$$f(t) = t + u_2(t)(3 - t) + u_3(t), \quad t \geq 0.$$

(b) (6 pts) Find all the equilibrium solutions of the equation $y' = \cos(y)$ and decide whether they are stable, unstable or semi-stable.

3. (10 pts) Let $y(t)$ be the solution of the following differential equation

$$y'' + 3y' + 2y = 0, y(0) = 2, y'(0) = -1.$$

(a) (5 pts) Find the solution $y(t)$.

(b) (5 pts) Find the maximum value of $y(t)$.

4. (10 pts)

(a) (5 pts) Derive a formula for $\mathcal{L}\{y'''(t)\}$ in terms of $\mathcal{L}\{y(t)\}, y(0), y'(0)$ and $y''(0)$.

(b) (5 pts) Use the definition of Laplace transform

$$\mathcal{L}\{y\} = \int_0^\infty e^{-st} f(t) dt$$

to show $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$. You are not allowed to use the conclusion from the Laplace Transform Table for this problem.

5. (12 pts) Use Laplace transforms to find the solution of the initial value problem

$$y'' + 4y = \delta_3(t) + u_3(t)$$

with $y(0) = 0, y'(0) = 0$.

6. (8 pts) A spherical raindrop evaporates at a rate proportional to its surface area with (positive) constant of proportionality k; i.e. the rate of change of the volume exactly equals $-k$ times the surface area. **Write a differential equation for the volume V of the drop as a function of time t.** (Hint: The surface area S of a sphere with radius r is $S = 4\pi r^2$. The volume of a spherical raindrop with radius r is $V = \frac{4}{3}\pi r^3$.)

7. (12 pts) The following are two different problems about spring-mass systems.

   (a) (6 pts) Consider a spring-mass system with $m = 2$ kg, $k = 1$ N/m, $\gamma = 1$ N·s/m. Suppose when $0 \leq t < 2$, there is no external force. When $2 \leq t < 4$, the external force is 2 N. When $t \geq 4$, the external force vanishes. Write down a differential equation (**without solving it**) for the displacement $u(t)$ of the spring from rest at time t. **you answer should include Heaviside step functions** $u_c(t)$.

   (b) (6 pts) The displacement $u(t)$ of a damped spring-mass system with constant external force satisfies $u'' + \gamma u' + 4u = 8$, where $\gamma > 0$. Find $\lim_{t \to +\infty} u(t)$.

8. (10 pts) Find the implicit solution to the first order differential equation

$$\frac{dy}{dx} - \frac{y}{2x} = e^x y^3, \quad y \neq 0.$$

Hint: Let $u = y^{-2}$.

9. (14 pts)

(a) (4 pts) Compute $h(t) = (f * g)(t)$, where $f(t) = t^2, g(t) = t$.

(b) (6 pts) Consider the function $H(s) = \dfrac{1}{s(s^2+1)}$. Find the inverse Laplace transform of $H(s)$ using the convolution.

(c) (4 pts) Find a function f such that $(f * 1)(t) = f(t) + 1$.