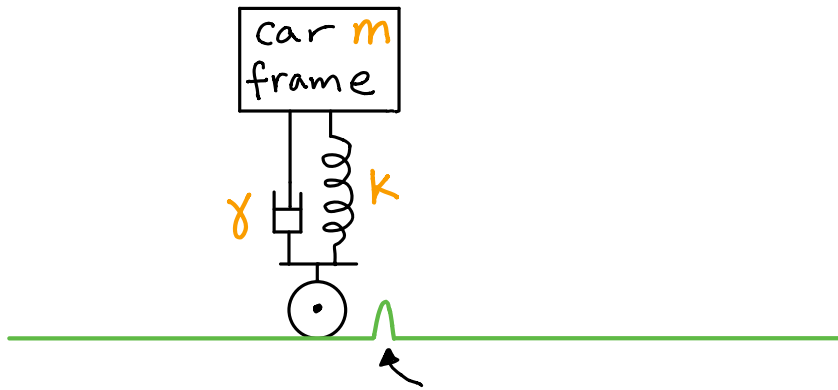


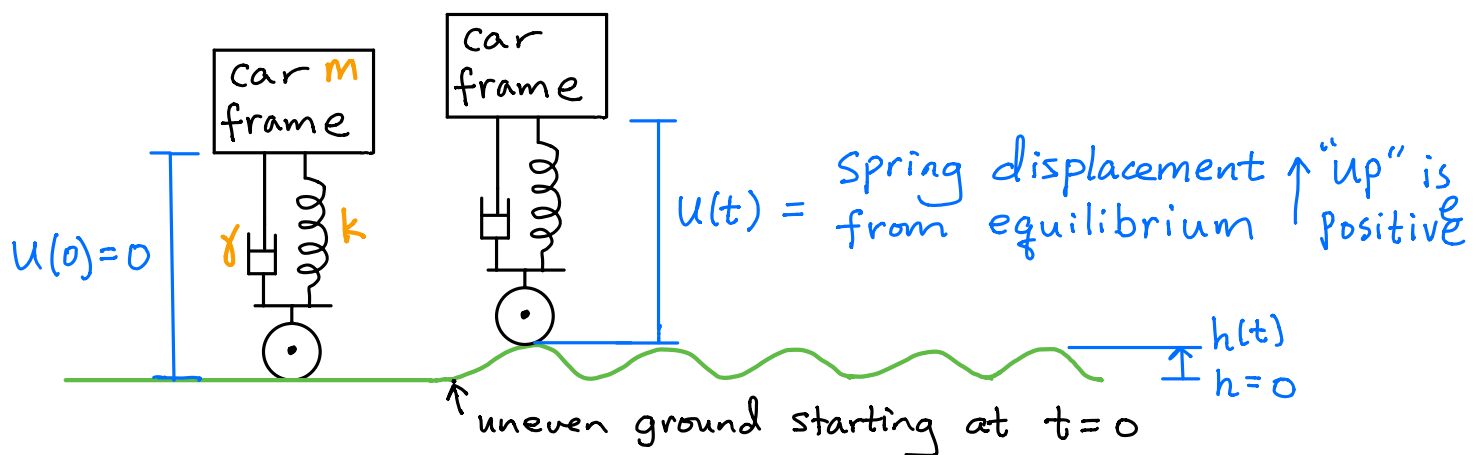
Forced Vibrations §3.8



sudden bump at $t=0$ gives nonzero initial condition

$$m u'' = -k u - \gamma u' \quad (u = \text{displacement from equil.})$$

$$m u'' + \gamma u' + k u = 0$$



$$m(\text{car frame acceleration}) = -k u - \gamma u'$$

$$m(u+h)'' = -k u - \gamma u'$$

$$m u'' + \gamma u' + k u = \underbrace{-m h''}_{\text{nonhomog. term: effective external driving force}}$$

There are many other examples of forced vibrations
(e.g. anytime sound wave hits an object)

$$m u'' + \gamma u' + k u = \underline{F(t)}$$

consider periodic driving force

$$F(t) = F_0 \cos(\omega t)$$

$$\text{or } F_0 \sin(\omega t)$$

$F_0, \omega > 0$
constants

Undamped Forced Vibration

$$m u'' + k u = F_0 \cos(\omega t)$$

Gen soln

$$\boxed{u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + u_p(t)} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

$$\boxed{\text{Case 1: } \omega \neq \omega_0} \quad (\text{Beats})$$

Ansatz: $u_p(t) = A \cos(\omega t) + B \sin(\omega t)$

$$F_0 \cos(\omega t) = m u_p'' + k u_p$$

$$= m(-A\omega^2 \cos(\omega t) - B\omega^2 \sin(\omega t)) + k(A \cos(\omega t) + B \sin(\omega t))$$

$$= (-m\omega^2 + k)A \cos(\omega t) + (-m\omega^2 + k)B \sin(\omega t)$$

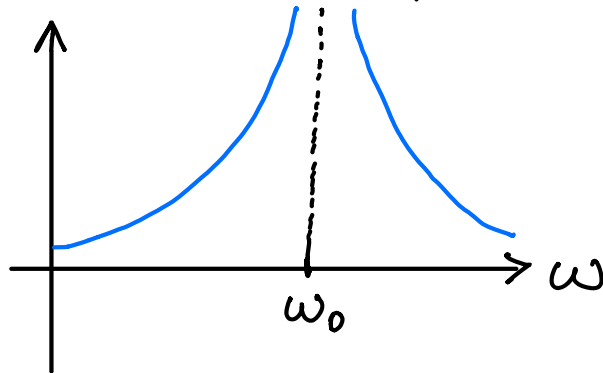
$$\Rightarrow A = \frac{F_0}{-m\omega^2 + k} = \frac{F_0}{m(\omega_0^2 - \omega^2)}, \quad B = 0$$

$C \cos(\omega_0 t - \delta)$

Gen soln: $U(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$

Sum of waves of different frequencies

Amplitude of $U(t) = \left| \frac{F_0}{m(\omega_0^2 - \omega^2)} \right| \rightarrow \infty$ when $\omega \rightarrow \omega_0$



Useful Trig identities :

$$\sin \theta \pm \sin \varphi = 2 \sin \frac{\theta \pm \varphi}{2} \cos \frac{\theta \mp \varphi}{2}$$

$$\cos \theta - \cos \varphi = -2 \sin \frac{\theta + \varphi}{2} \sin \frac{\theta - \varphi}{2}$$

$$\cos \theta + \cos \varphi = 2 \cos \frac{\theta + \varphi}{2} \cos \frac{\theta - \varphi}{2}$$

Example: Let $u(0) = 0$, $u'(0) = 0$

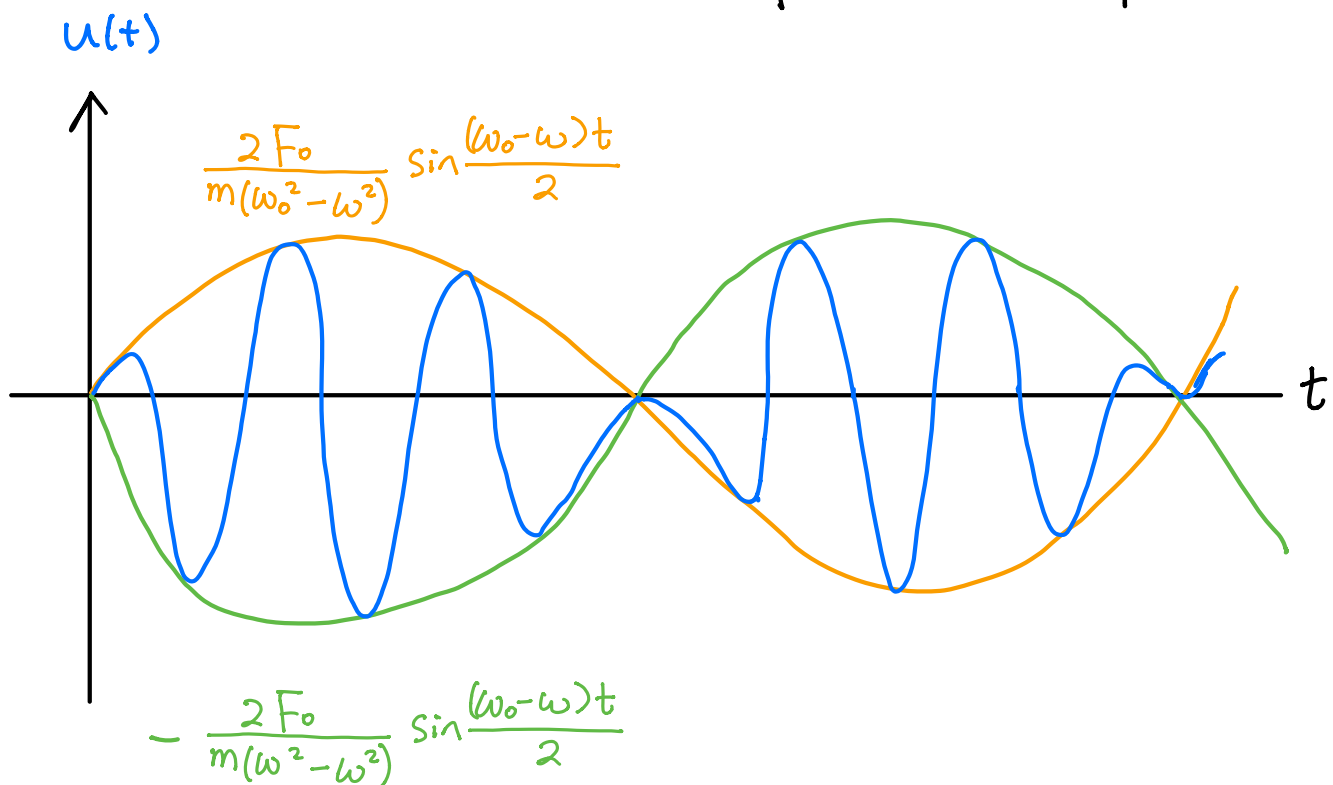
$$\Rightarrow C_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)}, \quad C_2 = 0$$

$$\begin{aligned}\Rightarrow u(t) &= \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos(\omega t) - \cos(\omega_0 t)) \\ &= \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\frac{(\omega_0 - \omega)t}{2} \sin\frac{(\omega_0 + \omega)t}{2}\end{aligned}$$

$$\left| \frac{\omega_0 - \omega}{2} \right| < \frac{\omega_0 + \omega}{2}$$

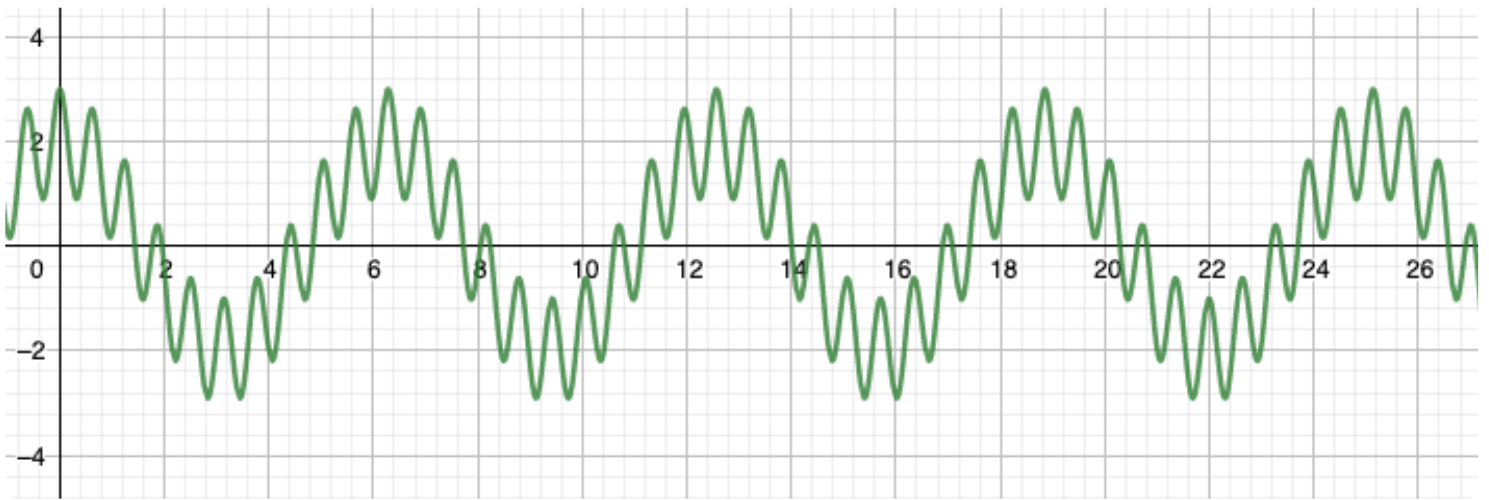
$$\Rightarrow \text{treat } \left| \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\frac{(\omega_0 - \omega)t}{2} \right|$$

as an amplitude envelope



Amplitude modulation

Example: $2\cos t + \cos(10t)$



Example: $u'' + 25u = \cos 3t$, $u(0) = 0$, $u'(0) = 0$

char eq: $r^2 + 25 = 0$

$$r^2 = -25$$

$$r = \pm 5i$$

$$u(t) = C_1 \cos 5t + C_2 \sin 5t + u_p(t)$$

Ansatz: $u_p(t) = A \cos 3t + B \sin 3t$

$$u_p''(t) = -9A \cos 3t - 9B \sin 3t$$

$$u_p'' + 25u_p = \cos 3t$$

$$(-9A \cos 3t - 9B \sin 3t) + 25(A \cos 3t + B \sin 3t) = \cos 3t$$

$$16A \cos 3t + 16B \sin 3t = \cos 3t$$

$$16A = 1, \quad 16B = 0$$

$$A = \frac{1}{16}, \quad B = 0$$

$$u(t) = C_1 \cos 5t + C_2 \sin 5t + \frac{1}{16} \cos 3t$$

$$0 = u(0) = C_1 + \frac{1}{16} \Rightarrow C_1 = -\frac{1}{16}$$

$$u'(t) = -5C_1 \sin 5t + 5C_2 \cos 5t - \frac{3}{16} \sin 3t$$

$$0 = u'(0) = 5C_2 \Rightarrow C_2 = 0$$

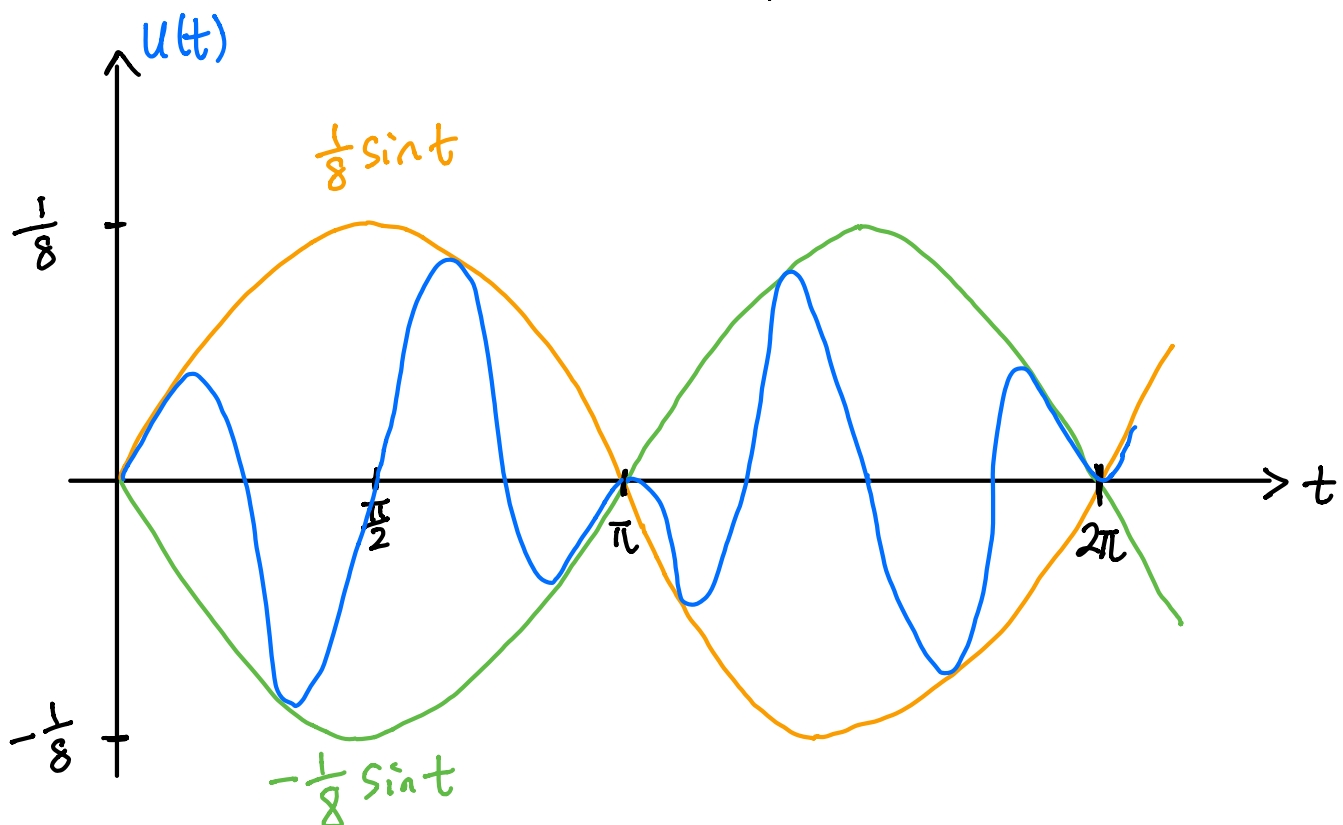
$$u(t) = -\frac{1}{16} \cos 5t + \frac{1}{16} \cos 3t$$

$$= -\frac{1}{16} (\cos 5t - \cos 3t)$$

$$= \frac{1}{8} \sin \frac{(5-3)t}{2} \sin \frac{(5+3)t}{2}$$

$$u(t) = \underbrace{\frac{1}{8} \sin t}_{\text{Amplitude}} \sin 4t$$

$$\text{Period of } \sin 4t = \frac{2\pi}{4} = \frac{\pi}{2}$$



Case 2 : $\omega = \omega_0$

(Resonance)

$$m u'' + k u = F \cos(\omega_0 t)$$

Ansatz : $u_p(t) = A t \cos(\omega_0 t) + B t \sin(\omega_0 t)$

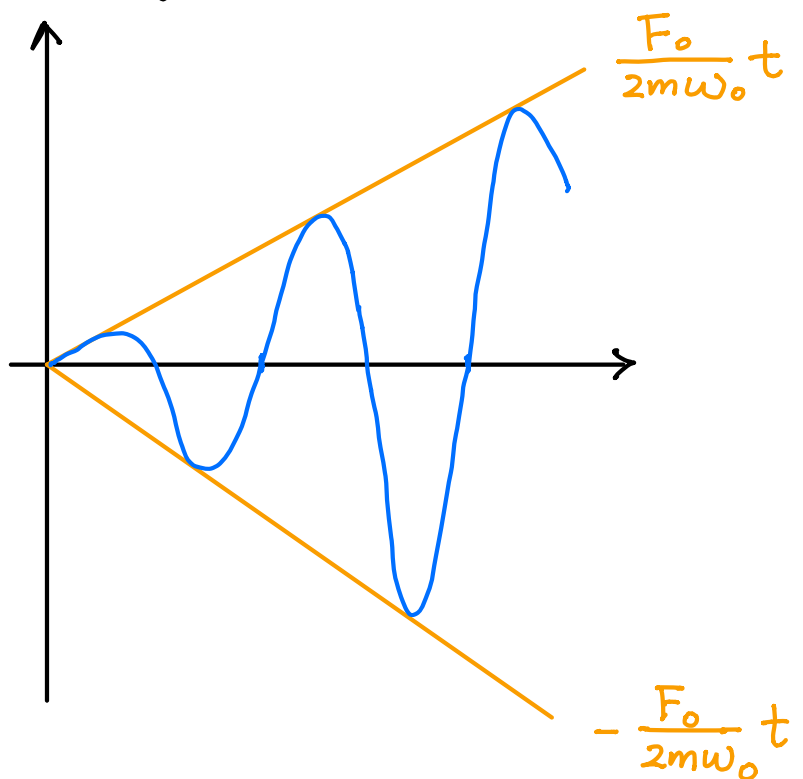
$$\begin{aligned} F \cos(\omega_0 t) &= m u_p'' + k u_p \\ &= m (u_p'' + \omega_0^2 u_p) \end{aligned}$$

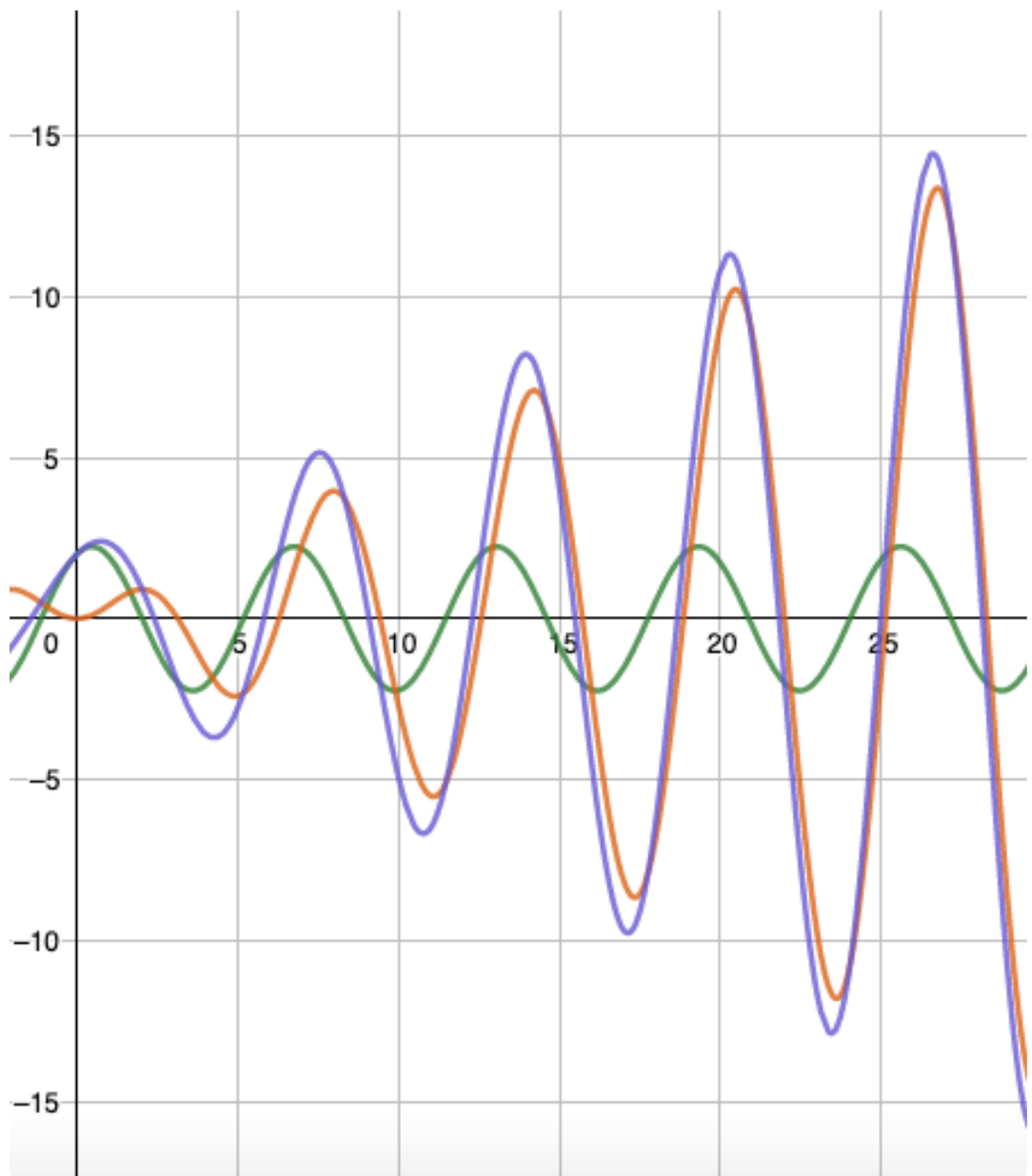
After simplification $\rightarrow \textcircled{=}$ $m (-2A \omega_0 \sin(\omega_0 t) + 2B \omega_0 \cos(\omega_0 t))$

$$\Rightarrow A = 0, \quad B = \frac{F_0}{2m \omega_0}$$

Gen soln : $u(t) = \underbrace{C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)}_{= C \cos(\omega_0 t - \delta)} + \frac{F_0}{2m \omega_0} t \sin(\omega_0 t)$

$$u_p(t) = \frac{F_0}{2m \omega_0} t \sin(\omega_0 t)$$





Purple : $2 \cos t + \sin t + \frac{1}{2} t \sin t$

Orange: $\frac{1}{2} t \sin t$

Green: $2 \cos t + \sin t$

Periodic driving force + damping

$$m u'' + \gamma u' + k u = F_0 \cos(\omega t)$$

$$\text{Gen soln: } u(t) = \underbrace{C_1 u_1(t) + C_2 u_2(t)} + u_p(t)$$

Gen soln to $m u'' + \gamma u' + k u = 0$
always decays, called "transient solution"

$$\text{Ansatz: } u_p(t) = C \cos \omega t + D \sin \omega t$$

$u_p(t)$ is called the "steady state soln"

$$u_p'(t) = -\omega C \sin \omega t + \omega D \cos \omega t$$

$$u_p''(t) = -\omega^2 C \cos \omega t + \omega^2 D \sin \omega t$$

$$F_0 \cos \omega t = m u_p'' + \gamma u_p' + k u_p$$

$$= [(k - m\omega^2)C + \gamma\omega D] \cos \omega t + [(k - m\omega^2)D - \gamma\omega C] \sin \omega t$$

$$\Rightarrow \left. \begin{aligned} (k - m\omega^2)C + \gamma\omega D &= F_0 \\ -\gamma\omega C + (k - m\omega^2)D &= 0 \end{aligned} \right\} \begin{aligned} C &= \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (\gamma\omega)^2} \\ D &= \frac{\gamma\omega F_0}{(k - m\omega^2)^2 + (\gamma\omega)^2} \end{aligned}$$

$$\text{Write } u_p = C \cos \omega t + D \sin \omega t = A \cos(\omega t - \delta)$$

$$\begin{aligned}
 A^2 = C^2 + D^2 &= \frac{(k - m\omega^2)^2 F_0^2 + (\gamma\omega)^2 F_0^2}{((k - m\omega^2)^2 + (\gamma\omega)^2)^2} \\
 &= \frac{F_0^2}{(k - m\omega^2)^2 + (\gamma\omega)^2} \\
 &= \frac{F_0^2}{m^2(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}
 \end{aligned}$$

$$\text{Amplitude } A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}}$$

 denominator
 always $\neq 0$ if $\gamma \neq 0$

Remark This calculation can also be done by
 Ansatz: $u_p = A \cos(\omega t - \delta) = \operatorname{Re} \underbrace{A e^{i(\omega t - \delta)}}_{= \tilde{u}_p}$

$$\operatorname{Re}(F_0 e^{i\omega t}) = \operatorname{Re}(m \tilde{u}_p'' + \gamma \tilde{u}_p' + k \tilde{u}_p)$$

Plot A vs ω

$$A(\omega) = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}}$$

$$\omega \rightarrow 0 \quad \Rightarrow \quad A \rightarrow \frac{F_0}{k}$$

$$\omega \rightarrow \infty \quad \Rightarrow \quad A \rightarrow 0$$

A has a max when $m^2(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2$ has a min

$$0 = \frac{d}{d\omega} [m^2(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]$$

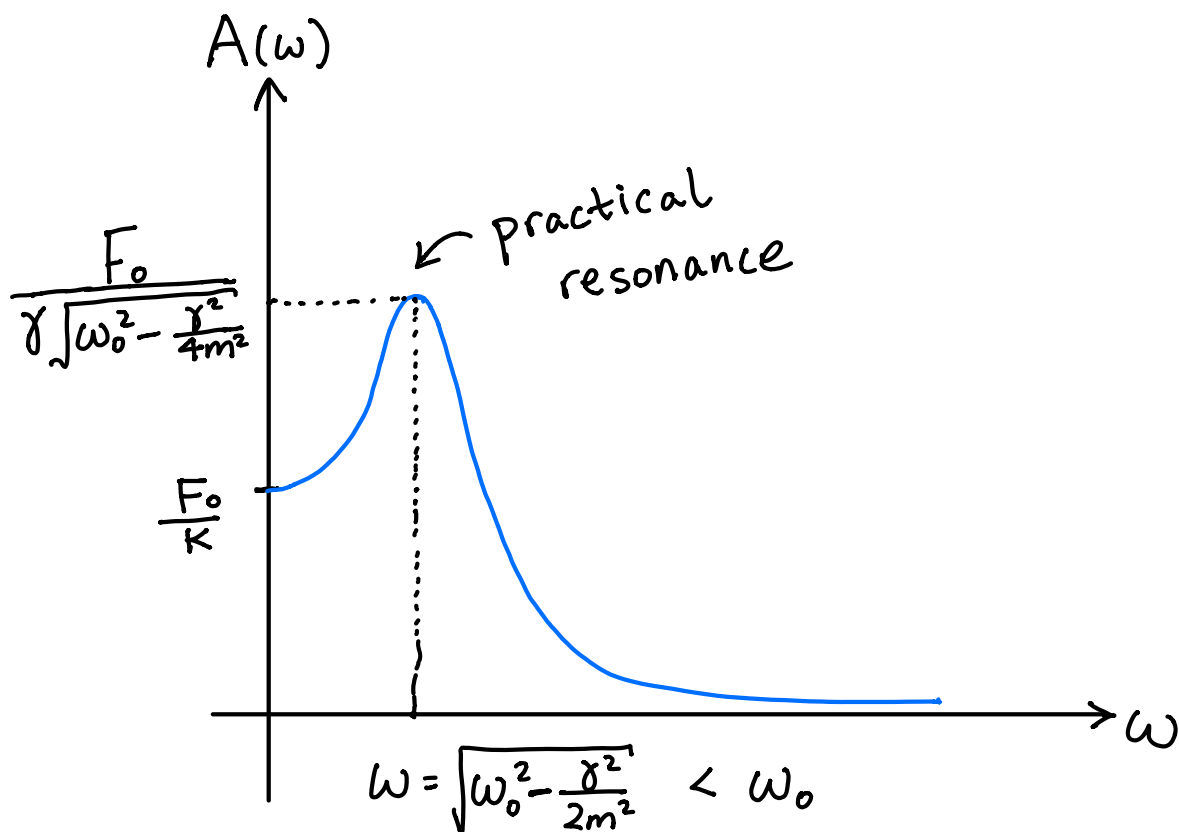
$$= (-2m^2(\omega_0^2 - \omega^2) + \gamma^2) 2\omega$$

$$\omega = 0, \quad \omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{2m^2}}$$

$$\text{real when } \omega_0^2 - \frac{\gamma^2}{2m^2} \geq 0$$

$$\Rightarrow \omega_0^2 - \left(\frac{\gamma}{2m}\right)^2 > \omega_0^2 - \frac{\gamma^2}{2m^2} \geq 0$$

$$\left(\begin{array}{l} mu'' + \gamma u' + ku = 0 \\ \text{underdamped} \end{array} \right)$$



By adjusting damping γ , can change resonance frequency.

Ex: $u'' + 4u' + 13u = 20 \cos(\omega t)$

char eq: $r^2 + 4r + 13 = 0$

$$r = -2 \pm 3i$$

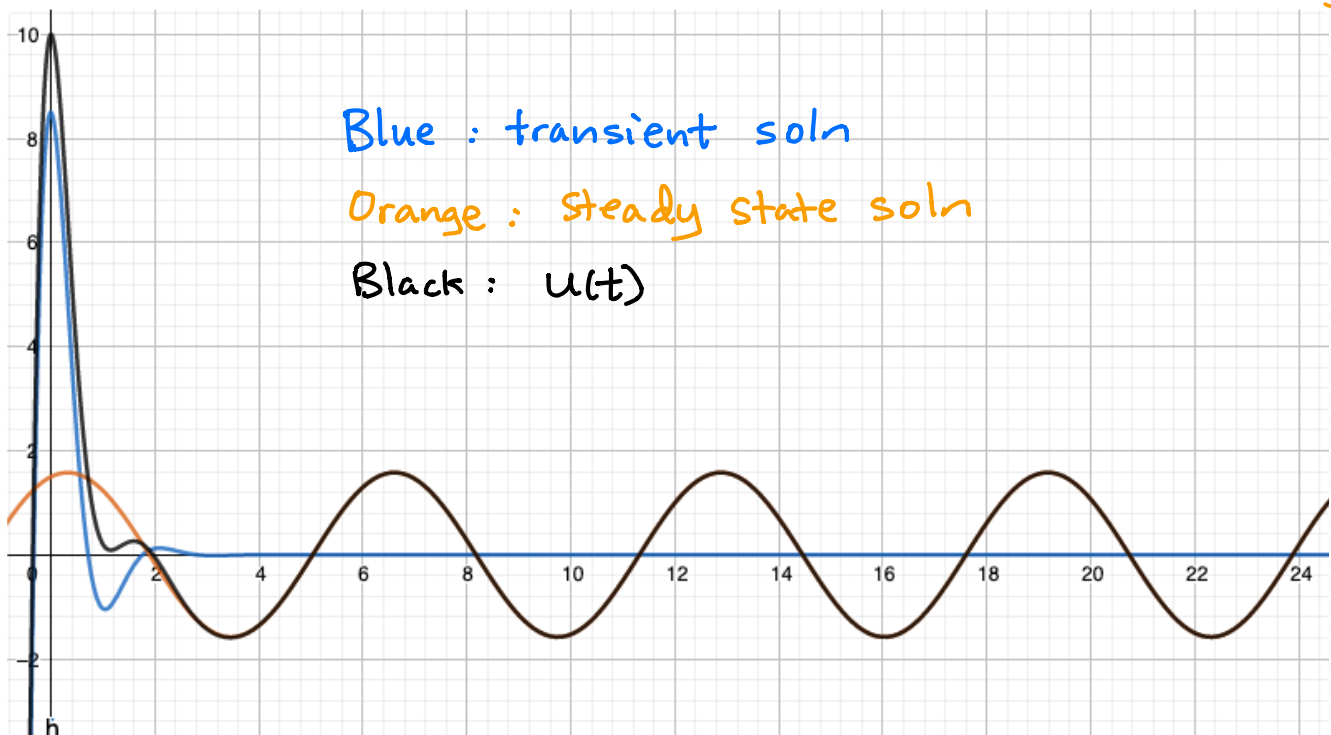
$$u = C_1 e^{-2t} \cos(3t) + C_2 e^{-2t} \sin(3t) + u_p(t)$$

Resonance happens when

$$\omega = \sqrt{13 - \frac{4^2}{2}} = \sqrt{5} \approx 2.24$$

(a) $u'' + 4u' + 13u = 20 \cos(t)$, $u(0) = 10$, $u'(0) = 0$

$$u(t) = 8.5 e^{-2t} \cos(3t) + 5.5 e^{-2t} \sin(3t) + \underbrace{\frac{3}{2} \cos t + \frac{1}{2} \sin t}_{=1.6 \cos(t-0.3)}$$



$$(b) \quad u'' + 4u' + 13u = 20 \cos(t) + 20 \cos(13t)$$

$$u_p(t) = \underbrace{\left(\frac{3}{2} \cos t + \frac{1}{2} \sin t \right)}_{= 1.6 \cos(t-0.3)} + \underbrace{\left(-\frac{3}{26} \cos(13t) + \frac{1}{26} \sin(13t) \right)}_{= 0.1 \cos(13t-3.4)}$$

