① Modeling

② Solving differential eqn

③ Interpretation

Population growth

$P(t)$ = total # of members in a population a time t

short term → * E. coli growth in petri dish/flusk of Luria broth

r = (exercise later) $\frac{1}{min}$

* Population growth of mice, fish

need more variables

* Money in bank with a fixed compounding interest rate 5% $\frac{1}{year}$

* Covid cases, $r_{WA} = 0.18 \frac{1}{day}$, $r_{US} = 0.36 \frac{1}{day}$

* radioactive decay

$$\frac{\Delta P}{\Delta t} = r P(t) \quad , \quad [r] = [\tfrac{1}{time}]$$

r ← growth rate

$$\boxed{\frac{dP}{dt} = r P(t)}$$

WA Covid : $P_{M20} = 1524$ (Mar 20), $P_{M21} = 1793$ (Mar 21)

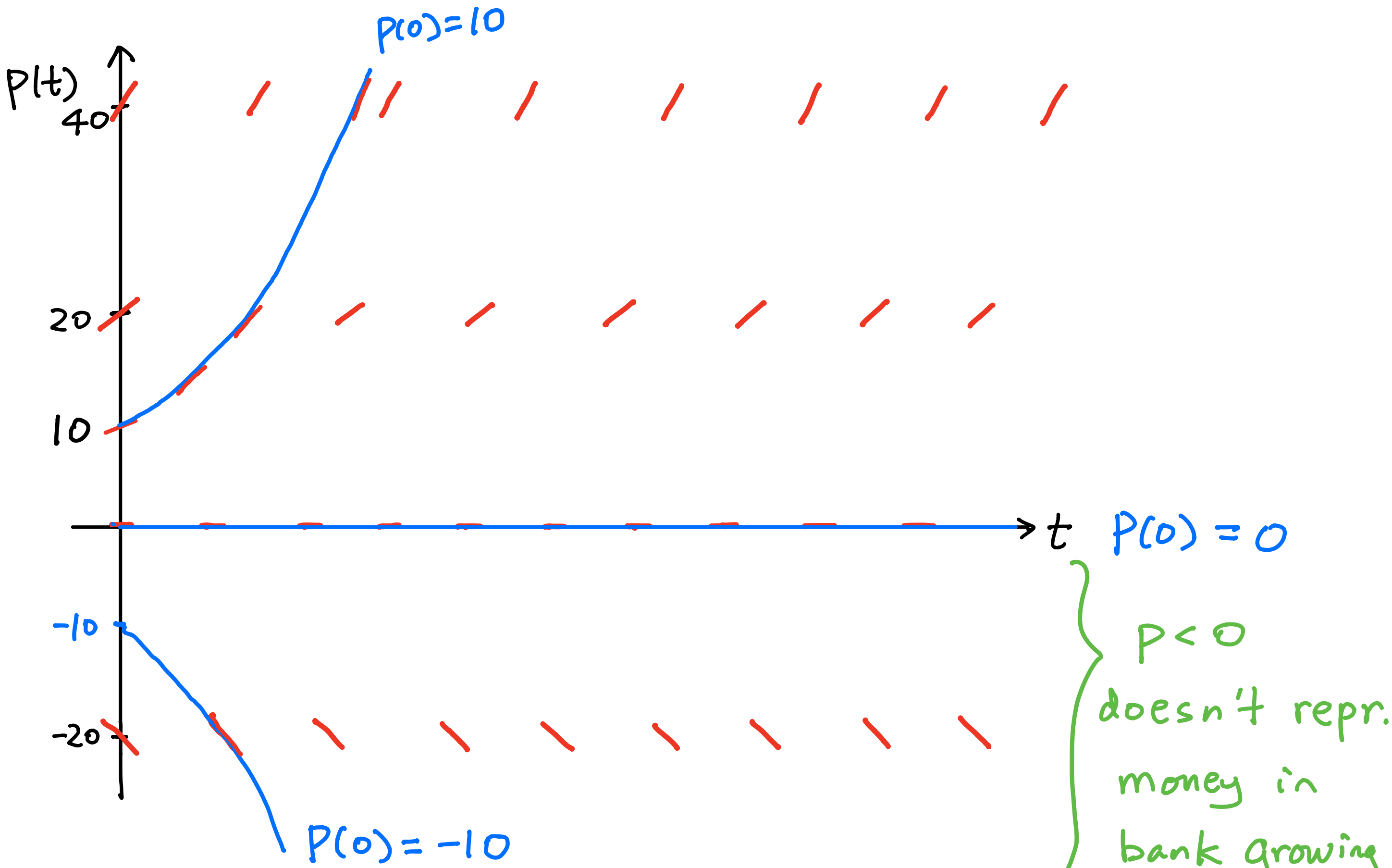
$$\Delta P = P_{M21} - P_{M20} = 269, \quad \Delta t = 1$$

$$269 = \frac{\Delta P}{\Delta t} = r\,P(t=M20) = r\,1524$$

$$r = 0.18 = 18\% \ \frac{1}{\text{day}}$$ (using just 1 data pt)

Direction field

$$\left.\begin{aligned} \frac{dP}{dt} &= 0.05\,P = \frac{P}{20} \\ P(0) &= P_0 \end{aligned}\right\}$$ initial value Problem (IVP)



$$\frac{dP}{dt} = f(P,t) = \frac{P}{20}$$

For each value of P, t, can compute $\frac{dP}{dt}$

$P = 40$, $t =$ anything $\Rightarrow \frac{dP}{dt} = 2$

$P = 20$, " $\Rightarrow \frac{dP}{dt} = 1$ $\Big\}$ $\left(\frac{dP}{dt} > 0, \text{ if } P > 0\right)$

$P = 0$, " $\Rightarrow \frac{dP}{dt} = 0 \Rightarrow$ $P(t) = 0$ for all t equilibrium solution

$P = -20$, " $\Rightarrow \frac{dP}{dt} = -1 \Big\} \left(\frac{dP}{dt} < 0, \text{ if } P < 0\right)$

* Each line segment is a tangent to one of the solution curves (aka integral curve)

* Each initial value $p(0)$ determines a solution curve $p(t)$

* If $P(0) > 0$, $\lim_{t\to\infty} P(t) = \infty$

* If $P(0) = 0$, " $= 0$

* If $P(0) < 0$, " $= -\infty$

end of lecture 1

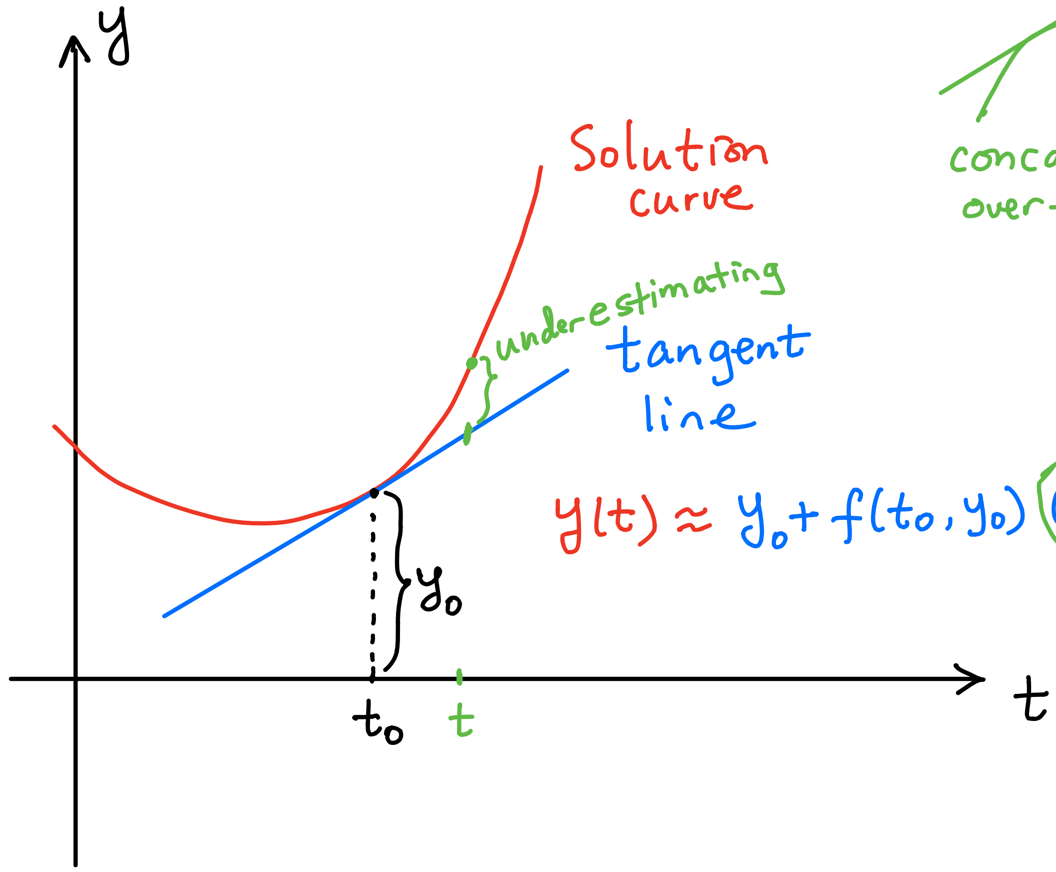
Euler's method §2.7

1st order diff. eqn: $\frac{dy}{dt} = f(t, y)$

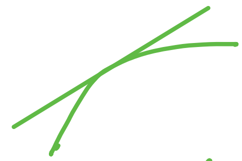
$y(t_0) = y_0$

Tangent line to the solution curve passing thru (t_0, y_0) has slope $f(t_0, y_0)$

Near (t_0, y_0)



$$y(t) \approx y_0 + f(t_0, y_0)\overbrace{(t - t_0)}^{h}$$



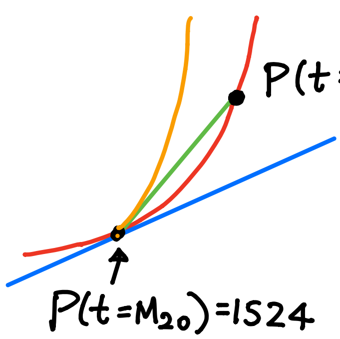
A line passing thru (t_0, y_0) w/ slope $f(t_0, y_0)$ has eqn:

$$f(t_0, y_0) = \frac{y - y_0}{t - t_0}$$

$$y - y_0 = f(t_0, y_0)(t - t_0)$$

$$y = y_0 + f(t_0, y_0)(t - t_0)$$

Aside: (WA covid example cont'd)



$$\frac{dP}{dt}(t=M20) \approx \frac{\Delta P}{\Delta t} = \frac{P_{M21} - P_{M20}}{M21 - M20} = 269$$

Euler's method example:

$$\boxed{\frac{dy}{dt} = y \quad , \qquad y(0) = 1}$$

$y'(0) = y(0) = 1$

step size $h = 0.5$ of Euler approx.

$$y(0.5) \approx y(0) + y'(0)\,0.5$$

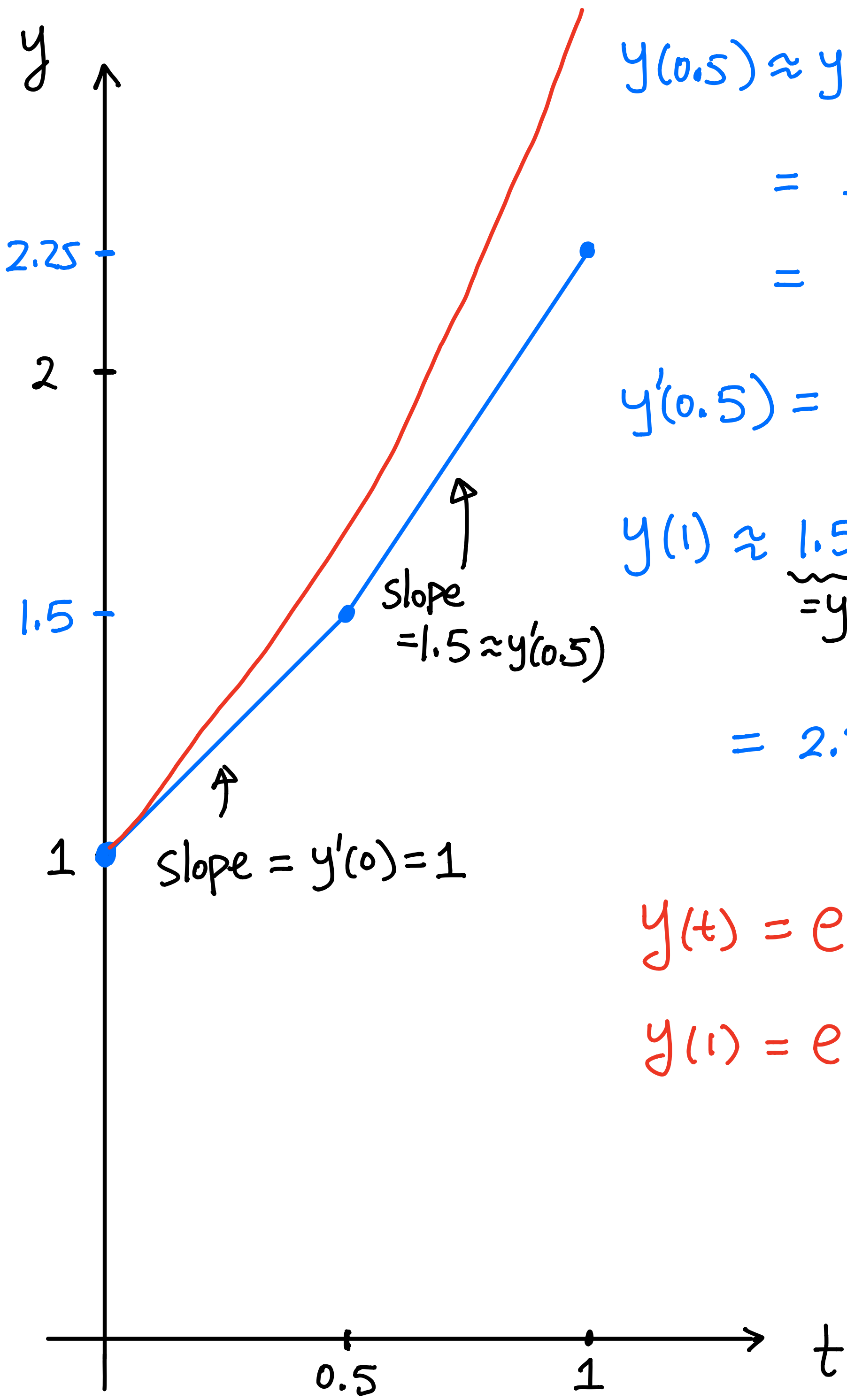
$$= 1 + (1)\,0.5$$

$$= 1.5$$

$$y'(0.5) = y(0.5) \approx 1.5$$

$$y(1) \approx \underbrace{1.5}_{=y(0.5)} + \underbrace{(1.5)}_{=y'(0.5)}\underbrace{(0.5)}_{=h}$$

$$= 2.25$$



$y(t) = e^t$

$y(1) = e = 2.71827\cdots$

Solving seperable eqn

Ex: $\frac{dP}{dt} = r\,P(t)$, $P(0) = P_0$

$$\int \frac{dP}{P} = \int r\,dt$$

(Assume $P \neq 0$
$P(t) = 0$ is the equilibrium soln)

$$\ln|P| = rt + C$$

$$|P| = e^{rt+C} = e^C e^{rt}$$

$$P = \pm e^C e^{rt}$$

$$\boxed{P(t) = A e^{rt}}$$

(Note $A = 0 \Rightarrow P(t) = 0$ equil. soln)

Check: $\frac{dP}{dt} = \frac{d}{dt}(Ae^{rt}) = Are^{rt} = r(Ae^{rt}) = rP$ ✓

$$P_0 = P(0) = Ae^{rt}\Big|_{t=0} = A$$

$$\boxed{P(t) = P_0 e^{rt}}$$

Exercise 1: e. coli in petri dish

doubling time is about 30 min, what is r?

(Hint and answer at the end of this note)

end of lecture 2

Ex : (WA covid-19 example cont'd)

$$\begin{cases} P(t=M20) = 1524 \\ P(t = M21) = 1793 \end{cases}$$

$$269 = \frac{\Delta P}{\Delta t} \approx \frac{dP}{dt}(t=M20) = r\,P(t=M20) = r\,1524$$

$$\frac{dP}{dt} = 0.18\,P \quad , \quad P(\underset{\substack{\uparrow \\ \text{March 20}}}{t=0}) = 1524$$

$$P(t) = 1524e^{0.18t}$$

$$P(t=\overset{\substack{\text{March 25} \\ \downarrow}}{5}) = 1524e^{(0.18)(5)} = 3748 > 2580 \text{ (actual)}$$

Issues with our model:

* $\frac{\Delta P}{\Delta t} > \frac{dP}{dt}$
* Only used one data pt, should take more existing data and find a best fit
* Most importantly(!), need to use a better model involving more variables that depends on t.

(Also simulation models and other sophisticated models + on-going research)

More on Euler's method (Optional)

$$y'(t) = f(t, y)$$

Sources of error

1) local truncation error from tangent line approximation

Function that can be represent by Taylor expansion

$$y(t) = \sum_{n=0}^{\infty} \frac{y^{(n)}(t_0)}{n!}(t-t_0)^n$$

$t = t_0 + h$

$$= \underbrace{y(t_0) + y'(t_0)\overset{h}{(t-t_0)}}_{} + \frac{y''(t_0)}{2!}(t-t_0)^2 + \frac{y'''(t_0)}{3!}(t-t_0)^3 + \cdots$$

local error

$$\left| y(t_0+h) - \big(y(t_0) + f(t_0, y_0)h \big) \right|$$

$$= \left| \frac{1}{2} y''(t_0) h^2 + \cdots \right|$$

Mean value theorem

$$= \left| \frac{1}{2} y''(\bar{t}_0) h^2 \right| \quad (\text{for some } t_0 \le \bar{t}_0 \le t_0 + h)$$

$$\le \frac{1}{2} M h^2 \quad (M = \max \{ |y''(t)| : t_0 \le t \le t_0 + h)$$

2) Global truncation error: include cumulative effect

$$|y(t_n) - \underbrace{y_n}_{\text{approx. value from the n'th step}}| \leq Kh$$

3) rounding off error

Improved Euler's method (Optional) Also see §8.2

$y'(t) = f(t, y)$

Recall Euler's method: $y_{n+1} = y_n + f(t_n, y_n)\, h$

Improved:

$$y_{n+1} = y_n + \frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2} h$$

replace y_{n+1} by $y_n + h f(t_n, y_n)$

$$\Rightarrow \boxed{y_{n+1} = y_n + \frac{f(t_n, y_n) + f\big(t_n + h,\ y_n + h f(t_n, y_n)\big)}{2} h}$$

local truncation error $\sim h^3$

Runge-Kutta (Optional see §8.3) : weighted at 4 pts in each interval

local truncation error $\sim h^5$

Population growth w/ harvesting

e.g. sustainable fishing

regular withdrawl of money in bank

$P(t)$ = # of members in the population

$$\frac{dP}{dt} = \underbrace{rP(t)}_{\text{growth w/o harvesting}} - \underbrace{450}_{\text{amount we harvest per unit time (e.g. per month)}}$$

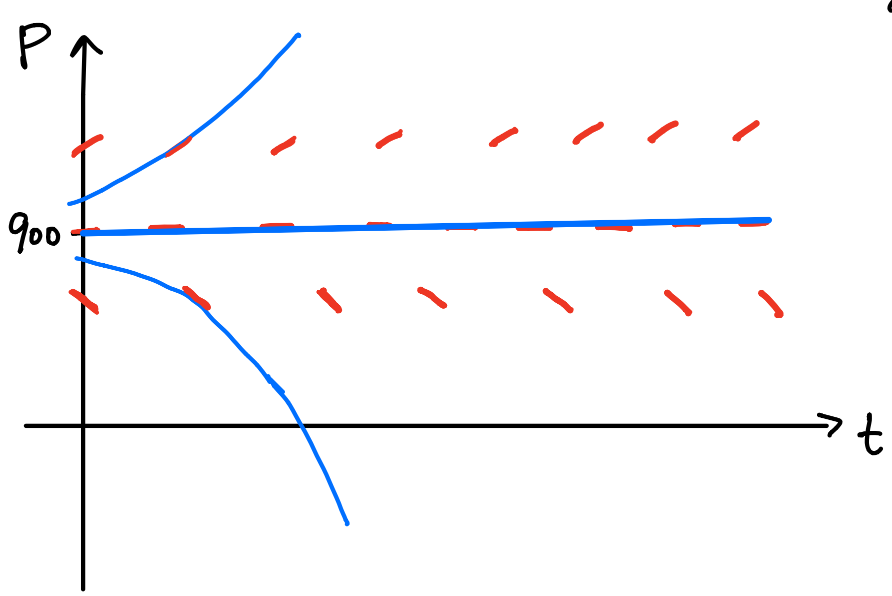
Suppose $r = 0.5 \frac{1}{\text{month}}$

$$\frac{dP}{dt} = 0.5P - 450$$

Note $\frac{dP}{dt} = 0$ when $P = 900$

If $P > 900$, $\frac{dP}{dt} > 0$

If $P < 900$, $\frac{dP}{dt} < 0$



$$\left.\begin{array}{l} \text{If } P(0) > 900, \quad P(t) \to \infty \quad \text{as } t \to \infty \\ \text{If } P(0) = 900, \quad P(t) = 900 \text{ for all } t \end{array}\right\} \text{sustainable}$$

If $P(0) < 900$, $P(t) \to -\infty$, as $t \to \infty$ (extinct due to overharversting)

(should be 0, not $-\infty$, for population)

Solve $\dfrac{dP}{dt} = 0.5P - 450$ (separable)

$$\int \frac{dP}{0.5P - 450} = \int dt$$

Exercise 2: finish this calculation to find $P(t)$

(Solution will be posted at the end of this note)

end of lecture 3

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below are solutions to exercises

Hint to exerise 1 (growth rate of e. coli)

$$P(t) = P_0 e^{rt} \quad \text{with} \quad P(0) = P_0$$

After 30 min, $P(30) = 2P_0$

Answer: $2P_0 = P(30) = P_0 e^{r30}$

$$e^{r30} = 2 \quad \Rightarrow \quad 30r = \ln 2$$

$$\Rightarrow \quad r = \frac{\ln 2}{30} \approx 0.023 \frac{1}{\text{min}}$$

Note that this is a change of base from e to 2.

If $P(t) = P_0 2^{\alpha t}$, then the doubling time is $\frac{1}{\alpha}$.

Note that $P_0 2^{\alpha t} = P_0 e^{(\ln 2)\alpha t} = P_0 e^{rt}$.

So $\quad (\ln 2)\alpha = r$

and the doubling time is

$$\frac{1}{\alpha} = \frac{\ln 2}{r} = 30 \text{ min}$$

so $\quad r = \frac{\ln 2}{30}$

Exerise 2

Solve $\frac{dP}{dt} = 0.5P - 450$

$$\int \frac{dP}{0.5P - 450} = \int dt$$

$u = 0.5P - 450$

$du = 0.5\, dP$, so $dP = \frac{du}{0.5}$

$$\int \frac{du}{0.5u} = \int dt$$

$$\frac{1}{0.5} \int \frac{du}{u} = \int dt$$

$$\frac{1}{0.5} \ln|u| = t + C$$

$$\ln|u| = 0.5t + C \leftarrow$$

here this is really 0.5C, but still just an arbitrary constant

$$0.5P - 450 = u = Ae^{0.5t}$$

$$0.5P = Ae^{0.5t} + 450$$

$$\boxed{P(t) = Ae^{0.5t} + 900}$$

note I changed A to 2A at some point, but since A is an arbitrary constant, 2A is also, so I just renamed 2A to be A.