

Spring 2022 Math 208 M Midterm 2

NAME (First,Last) :

STUDENT ID

UW email

- Please use the same name that appears in Canvas.
- **IMPORTANT:** Your exam will be scanned: **DO NOT** write within 1 cm of the edge. Make sure your writing is clear and dark enough.
- Write your NAME (first, last) on top of every odd page page of this exam.
- If you run out of space, continue your work on the back of the last page and indicate clearly on the problem page that you have done so.
- Unless stated otherwise, you **MUST** show your work and justify your answers.
- Your work needs to be neat and legible.

Problem 1 Let A be a 4×4 matrix with columns c_1, c_2, c_3, c_4 . Suppose that by performing a sequence of elementary operations you can reduce A to

$$B = \begin{pmatrix} 1 & 5 & -1 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

1. Find the rank of A . No justification necessary.

$$\text{rank}(A) = 3$$

2. Find a basis for $\text{row}(A)$, the row space of A .

$$\begin{pmatrix} 1 & 5 & -1 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

3. Find a basis for $\text{Null}(A)$, the null space of A . Show your work.

$$\begin{aligned} x_3 + 2x_4 &= 0 & \rightarrow x_3 &= -2x_4 \\ x_2 + x_3 - 2x_4 &= 0 & \rightarrow x_2 &= -x_3 + 2x_4 = 4x_4 \\ x_1 + 5x_2 - x_3 &= 0 & \rightarrow x_1 &= -5x_2 + x_3 = -22x_4 \end{aligned}$$

General solution of $Ax = 0$ in vector form is:

$$(-22x_4, 4x_4, -2x_4, x_4) = x_4(-22, 4, -2, 1)$$

Basis for $\text{Null}(A)$ is $(0, -4, -2, 1)$ (any non zero scalar multiple of this vector would do)

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4. Is c_1, c_2, c_4 a basis for $\text{col}(A)$, the column space of A ? Justify your answer.

Yes if we look at $\begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ This matrix is in reduced echelon form and it has 3

pivots so its columns are linearly independent
This means c_1, c_2, c_4 are 3 linearly independent vectors in $\text{col}(A)$ and since $\text{rank } A = 3$
They form a basis for $\text{col } A$

Consider $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$, $T(\vec{v}) = A\vec{v}$, where A is the matrix from the previous page.

1. Is T onto? Justify your answer.

No because $\text{rank}(A) = 3$ so
 $\text{col}(A)$ has dimension 3 and therefore
it is not equal to \mathbb{R}^4

2. Is T one to one? Justify your answer.

No by unifying theorem if T is not
onto it cannot be one to one

OR

No because $\text{Null}(A) \neq \{0\}$ (see part 3)
or consider that $\text{nullity}(A) = 4 - 3 = 1 \neq 0$

Problem 2 This problem has two unrelated parts.

Find the matrices for 2 different linear transformations T_1 and $T_2: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ both having the values $T_1((1, 0, 0)) = T_2((1, 0, 0)) = (2, -1)$ and $T_1((0, 1, 1)) = T_2((0, 1, 1)) = (1, 0)$ or explain why this is not possible.

The matrix M of a linear transformation from \mathbb{R}^3 to \mathbb{R}^2 has for columns $T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$ $T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)$ $T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$

We know that $T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ suppose $T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} a \\ b \end{pmatrix}$

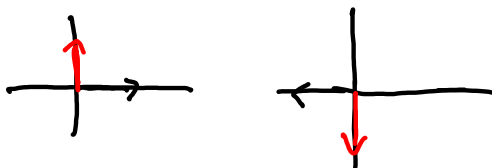
then $T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = T\left(\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1-a \\ -b \end{pmatrix}$

so any matrix of the form $\begin{bmatrix} 2 & a & 1-a \\ -1 & b & -b \end{bmatrix}$ would

work. We could have the matrix for T_1 , $A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$

and the matrix for T_2 $B = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ (many other answers are possible)

Find the matrix of the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that rotates a vector (x, y) 180 degrees. Show your work to explain how you found this matrix.



We want $T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$
 $T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

So $M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

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Problem 3 This problem has two unrelated parts.

1. Give an example of a 3×3 matrix A such that $\text{Null}(A) = \text{span} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right)$, and $\text{col}(A) = \text{span} \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right)$, or explain why this is not possible.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ a & b & c \\ 0 & 0 & 0 \end{bmatrix} \quad \text{This tells us } \text{col}(A) = \text{span} \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \\ \text{not all of } a, b, c = 0$$

$$\text{We also need } A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ so } a = 0$$

$$A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} : \quad \cdot 0$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{would work}$$

2. Give an example of a 3×3 invertible matrix A such that $\text{col}(A) = \text{span} \left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right)$, or explain why this is not possible.

$$\text{Not possible, if } \text{col } A = \text{span} \left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right)$$

$$\text{rank } A = 2 \text{ not } 3 \text{ so } A \text{ is not invertible}$$