Math 308L - Autumn 2017
Midterm 2
November 15, 2017

KEY

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
Total:	60	

- There are 5 problems on this exam. Be sure you have all 5 problems on your exam.
- The final answer must be left in exact form. Box your final answer.
- You are allowed the TI-30XIIS calculator. It is possible to complete the exam without a calculator.
- You are allowed a single sheet of 2-sided handwritten self-written notes.
- You must show your work to receive full credit. A correct answer with no supporting work will receive a zero.
- Use the backsides if you need extra space. Make a note of this if you do.
- Do not cheat. This exam should represent your own work. If you are caught cheating, I will report you to the Community Standards and Student Conduct office.

Conventions:

- I will often denote the zero vector by 0.
- When I define a variable, it is defined for that whole question. The A defined in Question 1 is the same for each part.
- I often use x to denote the vector $(x_1, x_2, \ldots, x_n)$. It should be clear from context.
- Sometimes I write vectors as a row and sometimes as a column. The following are the same to me.

$$(1,2,3) \quad \begin{bmatrix}1\\2\\3\end{bmatrix}.$$

- I write the evaluation of linear transforms in a few ways. The following are the same to me.

$$T(1,2,3) \quad T((1,2,3)) \quad T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right)$$

1. Answer the following parts:

(a) (6 points) Let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

i. What is A^{-1}?

Solution:

$$A^{-1} = \begin{bmatrix} 1 & -3 & 1/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1/3 \end{bmatrix}$$

ii. What is $\det(2 \cdot A^{-1})$?

Solution:

$$\det(2 \cdot A^{-1}) = 2^3 \det(A^{-1}) = 8/3.$$

(b) (6 points) (Tricky.) Let

$$B = \begin{bmatrix} 1 & 1 & 11 \\ -1 & 0 & 15 \\ 1 & 2 & 2017 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}.$$

It turns out that y is in the span of the first and second column of B and B is invertible. What is $B^{-1}y$? (Hint: Despite appearances, this is a quick computation.)

Solution: Denote the columns of B by b_1, b_2, b_3. Let $B^{-1}y = x$. Then $Bx = y$ and we are trying to find x. This amounts to solving a linear system. If $x = (x_1, x_2, x_3)$, then

$$x_1 b_1 + x_2 b_2 + x_3 b_3 = y.$$

But we know that y is in the span of b_1 and b_2 so $x_3 = 0$ and we are left with

$$x_1 b_1 + x_2 b_2 = y.$$

This yields the much easier linear system

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 0 & -2 \\ 1 & 2 & 0 \end{array}\right]$$

which has solutions $x_1 = 2$ and $x_2 = -1$. So altogether, we have

$$(x_1, x_2, x_3) = (2, -1, 0).$$

2. Give an example of each of the following. If it is not possible, write "NOT POSSIBLE".

(a) (3 points) Give an example of 2 linear transforms $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T \circ S : \mathbb{R}^2 \to \mathbb{R}^2$ is invertible.

Solution: Let $T(x, y, z) = (x, y)$ and $S(x, y) = (x, y, 0)$. Then $(T \circ S)(x, y) = (x, y)$ which is the identity transform which is invertible.

(b) (3 points) Give an example of a basis for $\mathbb{R}^3$ such that every basis element lies in the plane $x+y+z = 0$.

Solution: NOT POSSIBLE. The set $x + y + z = 0$ is a 2-dimensional subspace. There is no basis of $\mathbb{R}^3$ that lie in a 2-dimensional subspace.

(c) (3 points) Give an example of two different matrices A and B such that $\text{col}(A) = \text{col}(B)$ and $\text{null}(A) = \text{null}(B)$.

Solution: Pick your favorite natural number n. Let $A = I_n$ and $B = 2 \cdot I_n$. Then $\text{col}(A) = \text{col}(B) = \mathbb{R}^n$ and $\text{null}(A) = \text{null}(B) = \{0\}$.

(d) (3 points) Give an example of two 2×2 matrices A and B such that $\det(A+B) \neq \det(A)+\det(B)$.

Solution: Let $A = I_2$ and $B = I_2$. Then

$$\det(A + B) = 4 \quad \text{and} \quad \det(A) = \det(B) = 1.$$

3. Let $v = (1, 1, -1)$ and $L_v = \text{span}(\{v\})$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transform that is the projection onto L_v. This tells us 2 things about T:

- $T(x) = x$ if $x \in L_v$,
- $T(x) = 0$ if x is orthogonal to v (so if $x \cdot v = 0$).

There exists a matrix A such that $T(x) = Ax$. The goal of this problem is to determine A.

(a) (4 points) Give a basis for $\mathbb{R}^3$ that contains v and 2 vectors orthogonal to v. (Hint: Recall that $(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1b_1 + a_2b_2 + a_3b_3$.)

Solution: There are two methods to find 2 vectors orthogonal to v.

- Eyeball it.
- Find 2 particular solutions to $v \cdot (x, y, z) = x + y - z = 0$.

We find that $(1, 0, 1)$ and $(0, 1, 1)$ are distinct vector orthogonal to v. A basis for $\mathbb{R}^3$ is then given by

$$\{v, (1, 0, 1), (0, 1, 1)\}.$$

(b) (4 points) Answer the following questions about A.

i. Give a basis for null(A).

Solution: The null space of A is the kernel of T. We see that this is spanned by $\{(1, 0, 1), (0, 1, 1)\}$.

ii. Give a basis for col(A).

Solution: We can see that T sends everything to L_v so a basis for col(A) = range(T) is $\{v\}$.

iii. What is the rank of A?

Solution: From the last part, we can see that the rank is 1.

iv. What is det(A)?

Solution: The determinant of A is zero as T is not invertible.

(c) (4 points) What is A? You may express A as a product of matrices and their inverses.

Solution: From past worksheets and lectures, we know that if $\{u_1, \ldots, u_n\}$ is a basis for $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is defined so that $T(u_i) = v_i$ for $i = 1, \ldots, n$ then the corresponding matrix for T is given by VU^{-1}, where $V = [v_i]$ and $U = [u_i]$.
In this case, we have that $T(v) = v$ and $T(1, 0, 1) = (0, 0, 0)$ and $T(0, 1, 1) = (0, 0, 0)$. The corresponding matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}^{-1}.$$

4. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transform defined by $T(x) = Ax$, where A and its reduced echelon form are defined as follows:

$$A = \begin{bmatrix} 1 & 2 & -1 & -3 \\ 2 & 4 & 0 & -4 \\ 3 & 6 & -1 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

To save time when writing the solutions, let's denote the columns of A by a_1, a_2, a_3, a_4.

(a) (3 points) What is a basis for $\text{row}(A)$?

Solution: A basis for $\text{row}(A)$ is given by the 2 nonzero rows of B.

(b) (3 points) What a basis for the range of T?

Solution: A basis for the range is given by $\{a_1, a_3\}$.

(c) (3 points) Write the columns of A corresponding to free variables as a linear combination of pivot columns of A.

Solution: Relations among the columns of A are exactly the relations among the columns of B. This means

$$a_2 = 2a_1 \quad \text{and} \quad a_4 = -2a_1 + a_3.$$

(d) (3 points) What is a basis for $\ker(T)$?

Solution: The general solution to A is $x = s_1(-2, 1, 0, 0) + s_2(2, 0, -1, 1)$. This means that a basis for $\ker(T)$ is $\{(-2, 1, 0, 0), (2, 0, -1, 1)\}$.

5. Let A and B be equivalent matrices given by

$$A = \begin{bmatrix} 2 & 4 & -1 & -2 \\ -1 & -3 & -1 & 0 \\ 1 & 1 & 2 & 2 \\ 2 & 6 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

Let a_1, a_2, a_3, a_4 be the columns of A. Let $S = \text{span}(\{a_1, a_2\})$ and $T = \text{span}(\{a_3, a_4\})$.

(a) (2 points) What is $\dim(\text{span}(\{a_1, a_2, a_3, a_4\}))$?

Solution: This is asking for the rank of A which is 3 because there are 3 nonzero rows of B.

(b) (2 points) What is a basis for null(A)?

Solution: The general solution to A is $x = s_1(-1/2, 1/2, -1, 1)$. This means that a basis for null(A) is $\{(-1/2, 1/2, -1, 1)\}$.

(c) (2 points) Denote that intersection of S and T by $S \cap T$. This is the subspace of vectors that are in $\text{span}(\{a_1, a_2\})$ **and** in $\text{span}(\{a_3, a_4\})$. What is $\dim(S \cap T)$?

Solution: We can see that S and T are distinct 2-dimensional spaces in col(A) which is a 3-dimensional space. This means that the intersection is 1-dimensional. This is the more geometric idea. See the next answer for the more algebraic one.

(d) (6 points) (Hard.) What is a basis for $S \cap T$?

Solution: We will investigate what it means for a vector to be in $S \cap T$. Suppose $v \in S \cap T$. This means that we can write v as a linear combination of a_1, a_2 and as a linear combination of a_3, a_4. So there exists scalar c_1, c_2, c_3, c_4 such that

$$v = c_1 a_1 + c_2 a_2 = c_3 a_3 + c_4 a_4.$$

The goal is to determine the constraints on c_1, c_2, c_3, c_4. By rearranging, this means that

$$c_1 a_1 + c_2 a_2 - c_3 a_3 - c_4 a_4 = 0.$$

This means that $(c_1, c_2, -c_3, -c_4) \in \text{null}(A)$. Then $(c_1, c_2, -c_3, -c_4) = s_1(-1/2, 1/2, -1, 1)$ for some s_1. So

$$v = -1/2 s_1 a_1 + 1/2 s_1 a_2 \tag{1}$$

for some s_1. The equation (1) characterizes all $v \in S \cap T$. We can see that it is a 1-dimensional space. A basis is obtained by plugging in any nonzero s_1 into (1) so a basis for $S \cap T$ is $\{a_1 - a_2\}$.