Math 308 Midterm #2, Autumn 2017

Name:

ID#:

Signature:

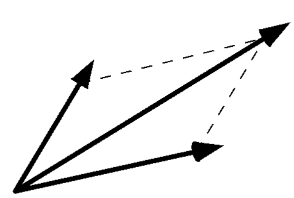
All work on this exam is my own.

Instructions.

- You are allowed a calculator and notesheet (handwritten, two-sided). Hand in your notesheet with your exam.
- Other notes, devices, etc are not allowed.
- Unless the problem says otherwise, **show your work** (including row operations if you row-reduce a matrix) and/or **explain your reasoning**. You may refer to any theorems, facts, etc, from class.
- All the questions can be solved using (at most) simple arithmetic. (If you find yourself doing complicated calculations, there might be an easier solution...)

1	/20
2	/25
3	/20
4	/20
5	/5

Good luck!



(1) (a) Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 0 & 2 & 0 \\ -1 & 1 & -3 \end{bmatrix}$. Compute A^{-1}, showing all work. [10 points]

Solution. By row-reducing:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 4 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ -1 & 1 & -3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -3 & 0 & -4 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & \frac{1}{2} & -4 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

Therefore $A^{-1} = \begin{bmatrix} -3 & \frac{1}{2} & -4 \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

(b) Let $L \subseteq \mathbb{R}^3$ be the line through the origin spanned by $\vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

Find linear equations that define L.
(That is, find a system of equations with solution set L.) [10 points]

Solution. Using the method from class, we need to find a basis for null($\begin{bmatrix} 1 & 1 & 3 \end{bmatrix}$), corresponding to the single equation $a + b + 3c = 0$. This is in echelon form already, with two free variables. So, setting free parameters $b = t_1$ and $c = t_2$, the nullspace consists of vectors

$$\vec{\mathbf{a}} = t_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

Finally, we interpret the two basis vectors as giving coefficients of equations in x, y, z. So, the equations are

$$\begin{aligned} -x_1 + x_2 \qquad &= 0 \\ -3x_1 \qquad + x_3 &= 0. \end{aligned}$$

Note that the vector $\vec{\mathbf{v}}$ is a solution (and the solution set is L).

(2) Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the transformation $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$. The matrix A, and an echelon form for A, are given below.

$$A = \begin{bmatrix} 3 & -2 & -1 & 3 \\ -1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 7 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Is T is one-to-one? Is T onto? [5 points each]

Solution. (Many explanations possible)
T is not onto because the echelon form has a zero row (non-pivot row).
T is not one-to-one because the echelon form has non-pivot columns.

(b) Give a basis for row(A) and a basis for col(A). [5 points each]

Solution. For row(A), we take the nonzero rows of the echelon form:

$$\text{basis for row}(A) : \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 9 \end{bmatrix} \right\}$$

For col(A), we take the columns of A corresponding to pivot columns of the echelon form:

$$\text{basis for col}(A) : \left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(c) What is nullity(A)? [5 points]

Solution. By part (b), A has rank 2. By the rank-nullity theorem, $\text{rank}(A) + \text{nullity}(A) = 4$, the number of columns of A. Therefore $\text{nullity}(A) = 2$.

(3) Let $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the coefficients of a quadratic polynomial, $f(t) = x_1 + x_2 t + x_3 t^2$.

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by $T(\vec{\mathbf{x}}) =$ the coefficients of $f'(t)$.

For example, $T\left(\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$, because $(1 + 3t + 2t^2)' = 3 + 4t$.

(a) Find a 3×3 matrix A such that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$. (The entries in A should be numbers. They should not involve t or x_1, x_2, x_3.) [10 pts]

Solution. We can find the matrix by finding $T(\vec{\mathbf{e}}_1), T(\vec{\mathbf{e}}_2), T(\vec{\mathbf{e}}_3)$.

For $T(\vec{\mathbf{e}}_1)$, the polynomial $f(t) = 1$, a constant, so $f'(t) = 0$ and $T(\vec{\mathbf{e}}_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

For $T(\vec{\mathbf{e}}_1)$, the polynomial $f(t) = t$, so $f'(t) = 1$ and $T(\vec{\mathbf{e}}_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

For $T(\vec{\mathbf{e}}_1)$, the polynomial $f(t) = t^2$, so $f'(t) = 2t$ and $T(\vec{\mathbf{e}}_1) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$.

Therefore $T(\vec{\mathbf{x}}) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \vec{\mathbf{x}}$.

(b) Find a basis for $\ker(T)$. If $\vec{\mathbf{x}} \in \ker(T)$, what does that tell us in terms of the polynomial $f(t)$? (Hint: it's a familiar fact from calculus.) [5 pts]

Solution. The kernel of T (i.e. the nullspace of A) consists of the solutions to the system of equations $x_2 = 0, 2x_3 = 0$. So, x_1 can be anything (it is a free variable), so the solutions are $\vec{\mathbf{x}} = s \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and a basis is just $\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \}$.

In terms of the polynomial $f(t)$, saying that $\vec{\mathbf{x}} \in \ker(T)$ means that the coefficients of $f'(t)$ are all zero, that is, $f'(t)$ is the zero function. From calculus, we know this happens only if $f(t) = c$, a constant.

(c) In terms of the polynomial $f(t)$, what is the meaning of the transformation $S(\vec{\mathbf{x}}) = A^2\vec{\mathbf{x}}$? Explain in a sentence. [5 pts]

Solution. The transformation multiplies by A, then by A again. (That is, it applies T twice in succession $S(\vec{\mathbf{x}}) = A \cdot A \cdot \vec{\mathbf{x}} = T(T(\vec{\mathbf{x}}))$.) Therefore $S(\vec{\mathbf{x}})$ gives the coefficients of the second derivative, $f''(t)$.

(4) Let A, B be $n \times m$ matrices. Let $S \subseteq \mathbb{R}^m$ be the set

$$S = \{\vec{\mathbf{x}} \in \mathbb{R}^m : A\vec{\mathbf{x}} = B\vec{\mathbf{x}}\}.$$

(a) Show that S is a subspace of $\mathbb{R}^m$. [10 pts]
(You may use either the definition, or any theorems or facts from class.)

Solution. By definition, a vector $\vec{\mathbf{x}}$ will be in S if and only if $A\vec{\mathbf{x}} = B\vec{\mathbf{x}}$. We have to check the three conditions for S to be a subspace:

- Is $\vec{\mathbf{0}} \in S$? Yes, because $A\vec{\mathbf{0}} = \vec{\mathbf{0}} = B\vec{\mathbf{0}}$.
- If $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in S$, does that guarantee $\vec{\mathbf{u}} + \vec{\mathbf{v}} \in S$? Yes, because
$$A(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = A\vec{\mathbf{u}} + A\vec{\mathbf{v}}$$
and this is equal to $B\vec{\mathbf{u}} + B\vec{\mathbf{v}}$ since both $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in S$. Therefore $A(\vec{\mathbf{u}} + \vec{\mathbf{v}})$ is equal to $B(\vec{\mathbf{u}} + \vec{\mathbf{v}})$, which means that $\vec{\mathbf{u}} + \vec{\mathbf{v}} \in S$.
- If $\vec{\mathbf{u}} \in S$ and $r \in \mathbb{R}$ is a scalar, does that guarantee $r\vec{\mathbf{u}} \in S$? Yes, because
$$A(r\vec{\mathbf{u}}) = rA\vec{\mathbf{u}}$$
and this is equal to $rB\vec{\mathbf{u}}$ since $\vec{\mathbf{u}} \in S$. Therefore $A(r\vec{\mathbf{u}}) = B(r\vec{\mathbf{u}})$, which means that $r\vec{\mathbf{u}} \in S$.

(b) Suppose $A = \begin{bmatrix} 3 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$. Find a basis for S. [10 pts]
(Hint: S is the set of all $\vec{\mathbf{x}}$ satisfying certain equations.)

Solution.
We have to find the vectors $\vec{\mathbf{x}}$ satisfying the equation $A\vec{\mathbf{x}} = B\vec{\mathbf{x}}$:

$$\begin{bmatrix} 3 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

We can solve this directly by writing out the two equations
$3x_1 + 2x_3 = 2x_1 + x_3$ and $x_1 + x_2 + x_3 = x_1 + 2x_3$, then simplifying and solving.

Many students found a slightly nicer way to do this, by rearranging the equation as $A\vec{\mathbf{x}} - B\vec{\mathbf{x}} = \vec{\mathbf{0}}$, or just $(A - B)\vec{\mathbf{x}} = \vec{\mathbf{0}}$:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{\mathbf{0}}.$$

This is in echelon form, with free variable x_3. After some algebra, we get

$$\vec{\mathbf{x}} = t \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},$$

so a basis for S is just $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(5) (a) Suppose A, B, D are square matrices and $A = B^{-1}DB$.
Simplify A^k to show $A^k = B^{-1}D^kB$, where k is a positive integer.
(If you wish, you can set $k = 3$). [5 pts]

Solution. The important point in this problem is that matrix multiplication is not commutative ($AB \neq BA$). So we can't 'distribute' the exponent: $(AB)^2$ means $AB \cdot AB$, and this is not the same as A^2B^2, which is $AABB$. So, we have to write out A^3 the long way:

$$\begin{aligned} A^3 = (B^{-1}DB)^3 &= (B^{-1}DB)(B^{-1}DB)(B^{-1}DB) \\ &= B^{-1}D\underbrace{BB^{-1}}_{=I}D\underbrace{BB^{-1}}_{=I}DB \\ &= B^{-1}DIDIDB \\ &= B^{-1}DDDB = B^{-1}D^3B. \end{aligned}$$

(b) (+3 bonus points)
Let $A = \begin{bmatrix} -2 & -10 \\ 2 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. **Note**: $A = B^{-1}DB$.

Using the formula in part (a), compute A^{2017}.
(Hint: Note that D is diagonal.)

Solution. Note: it's not possible to compute this with a calculator: the solution involves 2^{2017} and 3^{2017} which are close to 1000 digits ($3^{2017} = (3^2)^{1008.5} \approx 10^{1000}$). We can only compute it algebraically.

First, we need $B^{-1} = \frac{1}{5-4}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$ using the formula for the inverse of a 2×2 matrix.
Also, since D is diagonal, $D^{2017} = \begin{bmatrix} 2^{2017} & 0 \\ 0 & 3^{2017} \end{bmatrix}$.

Now we multiply the matrices:

$$\begin{aligned} A^{2017} = B^{-1}D^{2017}B &= \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2^{2017} & 0 \\ 0 & 3^{2017} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 5 \cdot 2^{2017} & -2 \cdot 3^{2017} \\ -2 \cdot 2^{2017} & 3^{2017} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 5 \cdot 2^{2017} - 4 \cdot 3^{2017} & 10 \cdot 2^{2017} - 10 \cdot 3^{2017} \\ -2 \cdot 2^{2017} + 2 \cdot 3^{2017} & -4 \cdot 2^{2017} + 5 \cdot 3^{2017} \end{bmatrix}. \end{aligned}$$

This is a closed expression for A^{2017}: pretty surprising (and useful!), since it would be quite difficult to directly compute

$$A^{2017} = \begin{bmatrix} -2 & -10 \\ 2 & 7 \end{bmatrix} \cdot \begin{bmatrix} -2 & -10 \\ 2 & 7 \end{bmatrix} \cdot \cdots \cdot \begin{bmatrix} -2 & -10 \\ 2 & 7 \end{bmatrix} \text{ (2017 times).}$$