Math 308L - Autumn 2017
Midterm 1
October 18, 2017

KEY

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
Total:	60	

- There are 5 problems on this exam. Be sure you have all 5 problems on your exam.
- The final answer must be left in exact form. Box your final answer.
- You are allowed the TI-30XIIS calculator. It is possible to complete the exam without a calculator.
- You are allowed a single sheet of 2-sided handwritten self-written notes.
- You must show your work to receive full credit. A correct answer with no supporting work will receive a zero.
- Use the backsides if you need extra space. Make a note of this if you do.
- Do not cheat. This exam should represent your own work. If you are caught cheating, I will report you to the Community Standards and Student Conduct office.

Conventions:

- I will often denote the zero vector by 0.
- When I define a variable, it is defined for that whole question. The A defined in Question 2 is the same for each part.
- I often use x to denote the vector $(x_1, x_2, \ldots, x_n)$. It should be clear from context.
- Sometimes I write vectors as a row and sometimes as a column. The following are the same to me.

$$(1,2,3) \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

- I write the evaluation of linear transforms in a few ways. The following are the same to me.

$$T(1,2,3) \quad T((1,2,3)) \quad T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$$

1. Consider the linear system of equations with the following augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 1 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

(a) (8 points) What is the general solution to this system of equations?

Solution: The reduced echelon form is:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right].$$

From this, we see that x_4 is a free variable which we'll set to s_1. The general solution is

$$(x_1, x_2, x_3, x_4) = (-1 + 2s_1, 1 - 3s_1, -s_1, s_1) = (-1, 1, 0, 0) + s_1(2, -3, -1, 1)$$

(b) (2 points) Write down 2 particular solutions to this system of equations.

Solution: By setting $s_1 = 0$, we obtain $(-1, 1, 0, 0)$. By setting $s_1 = 1$, we obtain $(1, -2, -1, 1)$.

(c) (2 points) What is the dimension of the solution space?

Solution: There is one free variable so the dimension of the solution space is 1.

2. Let A be the matrix

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ to be the linear transform defined by $T(x) = Ax$.

(a) (3 points) What is the general solution to $Ax = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$?

Solution: Notice that A is already in reduced echelon form. The free variable is x_3 which we'll set to s_1. The general solution is

$$(x_1, x_2, x_3, x_4) = (2 - 2s_1, 3 - 3s_1, s_1, 4) = (2, 3, 0, 4) + s_1(-2, -3, 1, 0)$$

(b) (3 points) Give 2 nontrivial solutions to $Ax = 0$.

Solution: The general solution is

$$(x_1, x_2, x_3, x_4) = s_1(-2, -3, 1, 0).$$

By setting $s_1 = 1$, we obtain $(-2, -3, 1, 0)$ and by setting $s_1 = -1$, we obtain $(2, 3, -1, 0)$.

(c) (3 points) Is T is one-to-one? If not, give a nontrivial solution to $T(x) = 0$.

Solution: No. We can take x to be one of the nontrivial solutions from the first part. For example, $x = (-2, -3, 1, 0)$.

(d) (3 points) Is T onto? If not, give a vector, b, such that $T(x) = b$ has no solution.

Solution: Yes. We can see that A has a pivot in each row.

3. Give an example of each of the following. If it is not possible, write "NOT POSSIBLE".

(a) (2 points) Give an example of a linear system of equations with more equations than variables and exactly one solution.

Solution:

$$x_1 = 1$$
$$x_1 = 1$$

(b) (2 points) Give an example of a linear system of equations with more variables than equations and exactly one solution.

Solution: NOT POSSIBLE. If there are more variables than equations, there must be a free variable. So there will either be no solutions or infinitely many.

(c) (2 points) Give an example of a set of 4 distinct vectors in $\mathbb{R}^3$ that do not span $\mathbb{R}^3$.

Solution: $(0, 0, 0), (1, 0, 0), (2, 0, 0), (3, 0, 0)$

(d) (2 points) Give an example of a linearly dependent set, S, of 3 vectors such that if we choose any pair of distinct vectors u, v in S, we have that u is not a scale multiple of v.

Solution: $(1, 0, 0), (0, 1, 0), (1, 1, 0)$.

(e) (2 points) Give an example of a linear transformation from $\mathbb{R}^3 \to \mathbb{R}^2$ that is onto.

Solution: Consider the linear transform T given by $T(x, y, z) = (x, y)$. The associated matrix has a pivot in each row.

(f) (2 points) Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(1, 0) = (2, 3)$ and $T(2, 0) = (3, 4)$.

Solution: NOT POSSIBLE. If T is a linear transform then $2T(1, 0) = T(2, 0)$ which is not the case here.

4. Let $S = \{u_1, u_2, u_3\}$ be a set of vectors in $\mathbb{R}^4$, where

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \quad u_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad u_3 = \begin{bmatrix} 4 \\ 1 \\ 3 \\ 5 \end{bmatrix}.$$

(a) (6 points) It turns out S is not linearly independent. Show this by writing u_3 as a linear combination of u_1 and u_2.

> **Solution:** We need to solve the linear system $x_1 u_1 + x_2 u_2 = u_3$. The solution is $x_1 = 2$ and $x_2 = 1$. So we have $2u_1 + u_2 = u_3$.

(b) (3 points) Write u_1 as a linear combination of u_2 and u_3.

> **Solution:** By using the last part, we have $u_1 = (u_2 - u_3)/2$.

(c) (3 points) It should be clear that S does not span $\mathbb{R}^4$. How many additional vectors are required to span $\mathbb{R}^4$? Be sure to briefly justify your answer.

> **Solution:** It is clear that $\{u_1, u_2\}$ is linearly independent but $\{u_1, u_2, u_3\}$ is not. So S spans a 2-dimensional space. Therefore, we need 2 additional vectors.

5. Let

$$u_1 = (1,0,0), u_2 = (1,1,0), u_3 = (1,1,1).$$

(a) (3 points) Express $(0,1,0)$ as a linear combination of u_1, u_2, u_3.

Solution: $(0,1,0) = -u_1 + u_2$

(b) (3 points) Express $(0,0,1)$ as a linear combination of u_1, u_2, u_3.

Solution: $(1,1,1) = -u_2 + u_3$.

(c) (3 points) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that $T(u_1) = (1,0), T(u_2) = (0,1)$, and $T(u_3) = (1,1)$. There exists a matrix A such that $T(x) = Ax$. What is A? (Hint: Use the first 2 parts and the fact that T is linear.)

Solution: We know that

$$T(0,1,0) = -T(u_1) + T(u_2) = (-1,1) \quad T(0,0,1) = -T(u_2) + T(u_3) = (1,0).$$

The matrix A is then given by writing the $T(1,0,0), T(0,1,0), T(0,0,1)$ as columns so

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(d) (3 points) Is T one-to-one? Is it onto? Briefly explain why. (Hint: This part does not require the correct solution to the 3rd part.)

Solution: The linear transform T is not one-to-one because it is going from a higher dimensional space to a lower dimensional space.
The linear transform T is onto because $(1,0)$ and $(0,1)$ are in the span.