Math 208 Final Section A

Mar 11th, 2024

Name______________________________ **Section Number** A

There are 9 problems, which are 50 points in total.

Instructions:

- (Academic honesty) **Sign your name below:**
  " I have not given or received any unauthorized help on this exam"
  [signature] ____________________
- You are allowed to use a summary sheet.
- No other resources are allowed (Internet, calculator, other humans, ...).
- **Answers must be justified or with necessary steps as required.**
- Please write down your initials **in each page**.
- **DO NOT WRITE ON THE BACK OF EACH PAGE.** Instead, use the last page for extra space.

Initials ____________________

Problem 1. (6 points) Answer the following questions.

(a)(3 points) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}5\\3\end{bmatrix}, \qquad T\left(\begin{bmatrix}2\\5\end{bmatrix}\right) = \begin{bmatrix}7\\4\end{bmatrix}.$$

Find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

(b)(3 points) State whether the following proposition is true or false. No explanation is necessary.

-(1 point) *Every* linear transformation $S : \mathbb{R}^4 \to \mathbb{R}^3$ is onto.

Your answer: ____________

-(1 point) *No* linear transformation $S : \mathbb{R}^4 \to \mathbb{R}^3$ is one-to-one.

Your answer: ____________

-(1 point) If A is an $n \times m$ matrix whose rows span $\mathbb{R}^m$, then the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is onto.

Your answer: ____________

Initials ____________________

Problem 2. (4 points) Consider the following vectors:

$$\mathbf{v}_1 = \begin{bmatrix} a_1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ -1 \\ a_2 - 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

Find all values of a_1 and a_2 such that $\mathbf{b}$ is NOT in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Explain your answer.

Initials ___________________

Problem 3. (6 points) Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 3 & 2 & -3 & -2 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

(1) (3 points) Find a basis for $\ker(T)$.

(2) (3 points) Find a linear transformation S that is one-to-one and satisfy

$$\text{range}(S) = \ker(T).$$

Initials ___________________

Problem 4. (7 points) For each of the following, find, if possible, an example and explain your example. OR explain why such an example cannot exist.

(1) (2 points) A 3×3 matrix A with $\det(A) = 0$ but with all entries nonzero.

(2) (3 points) A 3×4 matrix A with $\text{nullity}(A) = 2$ and $\text{row}(A) = \text{span}(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix})$.

(3) (2 points) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that is one-to-one but not onto.

Initials ___________________

Problem 5. (9 points) Suppose the characteristic polynomial of a $n \times n$ matrix A is

$$p_A(\lambda) = (-1-\lambda)\lambda(1-\lambda)(2-\lambda).$$

(1) (2 point) Find the value of n and find all eigenvalues of A.

(2) (1 point) Do the row vectors of A span $\mathbb{R}^n$? You do NOT need to explain why.

(3) (2 point) Find $\operatorname{rank}(A + 2I_n)$. Briefly explain why.

(4) (2 point) Find $\operatorname{rank}(A - I_n)$. Briefly explain why.

(5) (2 point) Is A diagonalizable? Briefly explain why.

Initials ___________________

Problem 6. (4 points) Consider the following matrix.

$$A = \begin{bmatrix} a & 3 & 0 \\ 1 & a & 1 \\ 0 & 1 & a \end{bmatrix}.$$

Find all values of a for which

$$\operatorname{rank}(A^3) < 3.$$

Initials ____________________

Problem 7. (7 points) Let A be the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}.$$

(1) (2 points) Compute the characteristic polynomials of A and find the eigenvalues. Show all of your work.

From the computation, we know A has two eigenvalues $\lambda_1 =$ ______ and $\lambda_2 =$ ______.

(2) (2 points) For λ_1 in question (1), find a basis for its eigenspace.

(3) (2 points) For λ_2 in question (1), find a basis for its eigenspace.

(4) (1 points) Is A diagonalizable? Explain why.

Initials ___________________

Problem 8. (3 points) Let A and B be the following matrices.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \qquad B = \begin{bmatrix} 2a & c & b+a \\ 2d & f & e+d \\ 2g & i & h+g \end{bmatrix}.$$

If $\det(A) = 1$, what is $\det(2B)$? Show all of your work.

Problem 9. (4 points) Let $A = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \mathbf{u}_4 \quad \mathbf{u}_5]$ with

$$\text{null}(A) = \text{span}\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 4 \\ 5 \\ 6 \end{bmatrix} \}.$$

Determine if $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_5$ are linearly independent or not. Explain why.