Math 208, Final Exam Name: ____________________

Signature: ____

Student ID #: ______________ Section #: I

- You are allowed a Ti-30x IIS Calculator and one 8.5×11 inch paper with handwritten notes on both sides. Other calculators, electronic devices (e.g. cell phones, laptops, etc.), notes, and books are **not** allowed.
- Some questions require you to explain answers: no explanation, no credit.
- Try to show your work on all questions: no work, no partial credit.
- You may use the back of the exam for scratch work: please submit any additional paper you use.
- Place a box around your answer to each question.
- Raise your hand if you have a question.

1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
T	/80

Good Luck!

(1) Let $A = \begin{pmatrix} 1 & 6 & 1 & 1 \\ -1 & 4 & 1 & 1 \\ 1 & 2 & -1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix}$

(a) (5pt) Compute the RREF of the augmented matrix $(A|b)$.

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} & 1 \\ 0 & 1 & 0 & \frac{1}{3} & 1 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 \end{pmatrix}$$

(b) (5pt) The set of all $x = (x_1\, x_2\, x_3\, x_4)^T \in \mathbb{R}^4$ satisfying $Ax = b$ is a line. Find the unique point in $\mathbb{R}^4$ where this line intersects the plane $x_2 = 0$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 3 \end{pmatrix}$$

(2) Give examples of the following, or explain why not possible. (2pts each)

(a) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ which is one-to-one, but not onto.
$T(x, y) = (x, y, 0)$

(b) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ which is one-to-one, but not onto.
Not possible—students may invoke the "unifying theorem".

(c) A diagonalizable matrix which is not invertible. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

(d) A matrix in RREF, w/ at least one entry $\neq 0$ or 1.. $\begin{pmatrix} 1 & 2 \end{pmatrix}$

(e) An $n \times n$ matrix of rank n in RREF, w/ at least one entry $\neq 0$ or 1. Not possible—full rank implies there is pivot in every row and column, and from this it follows that only the identity matrix is possible.

(3) Let $R = \frac{1}{7}\begin{pmatrix} 2 & 6 & 3 \\ -6 & 3 & -2 \\ -3 & -2 & 6 \end{pmatrix}$ (a) (5pt) True or false: is $R^{-1} = R^T$? Justify your answer. (Hint: you don't need to compute R^{-1} directly.)True—easiest to verify by computing the matrix product RR^T.

(b) (5pt) Find a nonzero vector $v \in \mathbb{R}^3$ such that $Rv = v$. We're looking for a nonzero vector in the nullspace of

$$R - I = \begin{pmatrix} -\frac{5}{7} & \frac{6}{7} & \frac{3}{7} \\ -\frac{6}{7} & -\frac{4}{7} & -\frac{2}{7} \\ -\frac{3}{7} & -\frac{2}{7} & -\frac{1}{7} \end{pmatrix}$$

Any nonzero multiple of

$$v = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

will do.

(4) Describe all values of t for which the given vectors *span* $\mathbb{R}^n$ (w/ $n = 2$ in (a), $n = 3$ in (b) and (c), and $n = 4$ in (d).) (2.5 pts each)

(a) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix}, \begin{pmatrix} t \\ 7 \end{pmatrix}$ All t except 7/2

(b) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ t \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$ All t except 5

(c) $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1+t \end{pmatrix}, \begin{pmatrix} 1 \\ t^2-3 \\ \cos(t) \end{pmatrix}$ All t except 0.

(d) $\begin{pmatrix} 0 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} t \\ t \\ 2 \\ 0 \end{pmatrix}$ No t whatsoever—3 vectors cannot span $\mathbb{R}^4$

(5) Consider the matrices $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\quad B = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

(a) (5pts) Calculate $A^{-1}B$ $\begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

(b) (5pts) Calculate the determinant of the following 4×4 matrix: 4

$$\left|\begin{array}{c|c} a_{11}B & a_{12}B \\ \hline a_{21}B & a_{22}B \end{array}\right| = \begin{vmatrix} 0 & 1 & 0 & 2 \\ 1 & 2 & 2 & 4 \\ 0 & 3 & 0 & 4 \\ 3 & 6 & 4 & 8 \end{vmatrix}$$

(The matrix in (b) is called the *Kronecker product* or *tensor product* $A \otimes B$.)

(6) Let A be a 3×3 matrix, and $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ its associated linear transformation, with the following eigenvalue/eigenvector pairs:

$$\lambda_1 = 2, \quad \mathbf{v}_1 = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix}^T$$
$$\lambda_2 = 1, \quad \mathbf{v}_2 = \begin{pmatrix} 3 & 4 & 0 \end{pmatrix}^T$$
$$\lambda_3 = 0, \quad \mathbf{v}_3 = \begin{pmatrix} 5 & 6 & 7 \end{pmatrix}^T$$

(a) (2.5pt) What is the rank of A? 2

(b) (2.5pt) What is the characteristic polynomial of A? $\lambda(\lambda - 1)(\lambda - 2) = \lambda^3 - 3\lambda^2 + 2\lambda$

(c) (5pt) Calculate $T\left(\begin{pmatrix} 12 & 16 & 7 \end{pmatrix}^T\right)$ $\begin{pmatrix} 8 \\ 12 \\ 0 \end{pmatrix}$

(7) Answer true/false, and justify answers: let $R = \begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $B = RAR^T$.

(a) T/F: R has 2 distinct, real eigenvaluesFalse—geometrically, one can see that rotation by any angle except multiples of π will never have a real eigenvector, hence no real eigenvalue

(b) T/F: B has 2 distinct, real eigenvalues True—notice that $RAR^T = A$ is a reflection matrix, with eigenvalues ± 1

(c) T/F: $\text{rank}(A) = \text{rank}(B) = \text{rank}(R) = 2$. True, since all three matrices are invertible—$R^{-1} = R^T$, and $A^{-1} = B^{-1} = A$.

(d) T/F: The linear system $Bx = b$ has no solution $x \in \mathbb{R}^2$ for some $b \in \mathbb{R}^2$ False—there is a unique solution $x = B^{-1}b$.

(e) T/F: Both standard basis vectors in $\mathbb{R}^2$ are contained in the subspace of $\mathbb{R}^2$ spanned by all eigenvectors of B. True—fixing an eigenbasis $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for B, we have $e_1 = (1/2)(v_1 + v_2)$ and $e_2 = (1/2)(v_1 - v_2)$

(8) The *Pell numbers* are a recursively-defined sequence of nonnegative integers. The first few terms of this sequence are

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, \ldots$$

If p_n is the n-th Pell number, we have $p_1 = 0$, $p_2 = 1$, and when $n \geq 2$ we define $p_{n+1} = p_{n-1} + 2p_n$. Using matrices, we have the recursive formula

$$\begin{pmatrix} p_n \\ p_{n+1} \end{pmatrix} = A \begin{pmatrix} p_{n-1} \\ p_n \end{pmatrix} = A^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \text{where} \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}.$$

The matrix A is diagonalizable—that is, $A = PDP^{-1}$ where P is an invertible 2×2 matrix and D is a 2×2 diagonal matrix.

(a) (6pts) Determine P and D that diagonalize A.

$$D = \begin{pmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ 1+\sqrt{2} & 1-\sqrt{2} \end{pmatrix}$$

(b) (4pts) Use (a) to find a simplified formula for p_n.

$$A^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1+\sqrt{2} & 1-\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} (1+\sqrt{2})^n & 0 \\ 0 & (1-\sqrt{2})^n \end{pmatrix} \cdot (-2\sqrt{2})^{-1} \begin{pmatrix} 1-\sqrt{2} & -1 \\ -(1+\sqrt{2}) & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= (-2\sqrt{2})^{-1} \begin{pmatrix} 1 & 1 \\ 1+\sqrt{2} & 1-\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} -(1+\sqrt{2})^n \\ (1-\sqrt{2})^n \end{pmatrix}$$

$$= \begin{pmatrix} \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} \\ \frac{(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}}{2\sqrt{2}} \end{pmatrix},$$

so $p_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$