Math 208, Final Exam Name: ______________________

Signature: ______________________________________

Student ID #: ______________ Section #: I

- You are allowed a Ti-30x IIS Calculator and one 8.5 $\times$ 11 inch paper with handwritten notes on both sides. Other calculators, electronic devices (e.g. cell phones, laptops, etc.), notes, and books are **not** allowed.
- Some questions require you to explain answers: no explanation, no credit.
- Try to show your work on all questions: no work, no partial credit.
- You may use the back of the exam for scratch work: please submit any additional paper you use.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Raise your hand if you have a question.

1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
T	/80

Good Luck!

(1) Let $A = \begin{pmatrix} 1 & 6 & 1 & 1 \\ -1 & 4 & 1 & 1 \\ 1 & 2 & -1 & 1 \end{pmatrix}$, $\quad b = \begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix}$

(a) (5pt) Compute the RREF of the augmented matrix $(A|b)$.

(b) (5pt) The set of all $x = (x_1\, x_2\, x_3\, x_4)^T \in \mathbb{R}^4$ satisfying $Ax = b$ is a line. Find the unique point in $\mathbb{R}^4$ where this line intersects the plane $x_2 = 0$.

(2) Give examples of the following, or explain why not possible. (2pts each)

(a) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ which is one-to-one, but not onto.

(b) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ which is one-to-one, but not onto.

(c) A diagonalizable matrix which is not invertible.

(d) A matrix in RREF, w/ at least one entry $\neq 0$ or 1..

(e) An $n \times n$ matrix of rank n in RREF, w/ at least one entry $\neq 0$ or 1.

(3) Let $R = \frac{1}{7}\begin{pmatrix} 2 & 6 & 3 \\ -6 & 3 & -2 \\ -3 & -2 & 6 \end{pmatrix}$ (a) (5pt) True or false: is $R^{-1} = R^T$? Justify your answer. (Hint: you don't need to compute R^{-1} directly.)

(b) (5pt) Find a nonzero vector $v \in \mathbb{R}^3$ such that $Rv = v$.

(4) Describe all values of t for which the given vectors *span* $\mathbb{R}^n$ (w/ $n = 2$ in (a), $n = 3$ in (b) and (c), and $n = 4$ in (d).) (2.5 pts each)

(a) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix}, \begin{pmatrix} t \\ 7 \end{pmatrix}$

(b) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ t \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$

(c) $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1+t \end{pmatrix}, \begin{pmatrix} 1 \\ t^2 - 3 \\ \cos(t) \end{pmatrix}$

(d) $\begin{pmatrix} 0 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} t \\ t \\ 2 \\ 0 \end{pmatrix}$

(5) Consider the matrices $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

(a) (5pts) Calculate $A^{-1}B$

(b) (5pts) Calculate the determinant of the following 4×4 matrix:

$$\left|\begin{array}{c|c} a_{11}B & a_{12}B \\ \hline a_{21}B & a_{22}B \end{array}\right| = \begin{vmatrix} 0 & 1 & 0 & 2 \\ 1 & 2 & 2 & 4 \\ 0 & 3 & 0 & 4 \\ 3 & 6 & 4 & 8 \end{vmatrix}$$

(The matrix in (b) is called the *Kronecker product* or *tensor product* $A \otimes B$.)

(6) Let A be a 3×3 matrix, and $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ its associated linear transformation, with the following eigenvalue/eigenvector pairs:

$$\begin{aligned} \lambda_1 &= 2, \quad \mathbf{v}_1 = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix}^T \\ \lambda_2 &= 1, \quad \mathbf{v}_2 = \begin{pmatrix} 3 & 4 & 0 \end{pmatrix}^T \\ \lambda_3 &= 0, \quad \mathbf{v}_3 = \begin{pmatrix} 5 & 6 & 7 \end{pmatrix}^T \end{aligned}$$

(a) (2.5pt) What is the rank of A?

(b) (2.5pt) What is the characteristic polynomial of A?

(c) (5pt) Calculate $T\left(\begin{pmatrix} 12 & 16 & 7 \end{pmatrix}^T\right)$

(7) Answer true/false, and justify answers: let $R = \begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$, $\quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $B = RAR^T$.

(a) T/F: R has 2 distinct, real eigenvalues

(b) T/F: B has 2 distinct, real eigenvalues

(c) T/F: $\text{rank}(A) = \text{rank}(B) = \text{rank}(R) = 2$.

(d) T/F: The linear system $Bx = b$ has no solution $x \in \mathbb{R}^2$ for some $b \in \mathbb{R}^2$

(e) T/F: Both standard basis vectors in $\mathbb{R}^2$ are contained in the subspace of $\mathbb{R}^2$ spanned by all eigenvectors of B.

(8) The *Pell numbers* are a recursively-defined sequence of nonnegative integers. The first few terms of this sequence are

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, \ldots$$

If p_n is the n-th Pell number, we have $p_1 = 0$, $p_2 = 1$, and when $n \geq 2$ we define $p_{n+1} = p_{n-1} + 2p_n$. Using matrices, we have the recursive formula

$$\begin{pmatrix} p_n \\ p_{n+1} \end{pmatrix} = A \begin{pmatrix} p_{n-1} \\ p_n \end{pmatrix} = A^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \text{where} \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}.$$

The matrix A is diagonalizable—that is, $A = PDP^{-1}$ where P is an invertible 2×2 matrix and D is a 2×2 diagonal matrix.

(a) (6pts) Determine P and D that diagonalize A.

(b) (4pts) Use (a) to find a simplified formula for p_n (Hint: this will involve some square roots and exponents depending on n.)