Math 208(A) Final

December 11, 2023

NAME:

Section: 10:30 or 11:30 (circle the one you are registered in)

UW EMAIL:

Instructions.[1]

- **Please write your initials in the top right hand corner of each page.**

- There are 10 problems on this exam. Each one is worth 10 points for a total of 100 points. There is also one bonus problem at the end worth 5 points.
- For each problem below give a carefully explained solution using the vocabulary and notation from class. A correct answer with no supporting work or explanation will receive a zero.
- Simplify your answers, collect all terms, and reduce all fractions. Put a box around your final answers.
- You are allowed a simple calculator and notesheet. Other notes, electronic devices, etc are not allowed. Take a few pencils from your pencil case out and put all other items away for the duration of the exam.
- All the questions can be solved using (at most) simple arithmetic. (If you find yourself doing complicated calculations, there might be an easier solution...)
- This exam is printed doubled sided. The last page is intentionally blank. You can use this as scratch paper or for more room for your solutions, but please label your work clearly if you intend for us to grade it.
- Raise your hand if you have any questions or spot a possible error.

Good luck!

[1] Test code: 3227

(1) Compute the determinant of the following matrix and briefly explain the steps you use for your computation. Put a box around the final answer.

$$\begin{bmatrix} 2 & 1 & 1 & 9 & -7 \\ 0 & 1 & -6 & 1 & 4 \\ 0 & 2 & 7 & 8 & -4 \\ 0 & 2 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}$$

(2) Find a 3×3 matrix A with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with $\lambda = 1$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ with $\lambda = 2$ and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ with $\lambda = 10$.

(3) Find a maximal size independent subset of the following vectors and briefly decribe the method you used to verify your claim.

$$\begin{bmatrix}1\\0\\1\\7\end{bmatrix}, \begin{bmatrix}0\\1\\0\\3\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\5\end{bmatrix}, \begin{bmatrix}2\\0\\-1\\0\end{bmatrix}$$

(4) Give an example of a matrix that satisfies the following properties if possible or say "not possible" and give a brief justification. (2pts each)

(a) A 2×2 matrix that is nonzero in every entry and $A = A^{-1}$.

(b) A 2×2 matrix that is nonzero in every entry and $A = 2A$.

(c) A 2×2 matrix that is nonzero in every entry and A and A^2 have different eigenvectors.

(d) A 2×2 matrix that is nonzero in every entry and $A = A^T$.

(e) A 2×2 matrix that is nonzero in every entry and $A^T = A^{-1}$.

(5) Let $\mathbf{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

(a) Draw the three vectors $\mathbf{v}, B\mathbf{v}, B^2\mathbf{v}$ in $\mathbb{R}^2$ labeling the axes and the endpoints of the vectors.

(b) Is there a value $k > 0$ such that $\mathbf{v}$ is an eigenvector of B^k? If so, what is the eigenvalue λ such that $B^k\mathbf{v} = \lambda\mathbf{v}$? If not, explain why not.

(6) Let A be the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 & 60 \\ 3 & -1 & 2 & 10 \\ 1 & 1 & -1 & 20 \end{bmatrix}.$$

(a) (2pts) What is the domain and codomain of the function $T(\mathbf{x}) = A\mathbf{x}$?

(b) (4pts) Give a basis for the range of T.

(c) (4pts) What is the dimension of the kernel of T? Justify your answer.

(7) We have finally gotten data from our intergalactic frequency detector! The observations are

$$f(-1) = 8,\ f(0) = 4,\ f(1) = 4,\ f(2) = 2$$

It might be a polynomial function of the input. Please help us figure it out. Is there a cubic polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ that would match the data we have so far? If so, help us figure out the coefficients.

(a) (2pts) What equations must a_0, a_1, a_2, a_3 satisfy?

(b) (6pts) Find all possible solutions to these equations.

(c) (2pts) Write down $f(x)$ and verify the data above is satisfied.

(8) If C is the change of basis matrix that takes the basis $\mathcal{B}_1$ to $\mathcal{B}_2$ for $\mathbb{R}^n$, is C always, sometimes, or never invertible? Justify your answer.

(9) (a) Find 3×3 invertible matrices A and B such that $Det(A+B) = Det(A) + Det(B)$.

(b) For $n \times n$ matrices, does $Det(A + B) = Det(A) + Det(B)$ always hold? Justify your answer.

(10) The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \ldots$ is an infinite sequence given recursively by the formula $F_{n+1} = F_n + F_{n-1}$ and the initial conditions $F_1 = 1$ and $F_2 = 1$.

(a) (3pts) Find a matrix A such that $A^{n-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$ and test it by computing $A\begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

(b) (3pts) What are the eigenvalues of A?

(c) (2pts) What are the dimensions of the eigenspaces of A?

(d) (2pts) Is A diagonalizable, invertible, both, or neither?

(11) (Bonus for 5pts) Let S be the set of all 5×5 matrices with entries in $\{0, 1\}$. What is the average of all determinants of matrices in S? (Hint: matrices with determinant 0 contribute 0 to the average.)

Blank page for scratch work or extra space. Please make a note on the problem page if you use this space for your answer.