MATH 208 B **Final Exam** **Winter 2022**

NAME:

UW ID:

Academic Honesty Statement: All work on this exam is my own.

Signature:

INSTRUCTIONS

1. This exam contains eleven (11) printed pages. Check that on your exam the bottom of the last page says **END OF EXAM**.
2. The points for each question are indicated at the beginning of each question.
3. **Show all your work**, unless the problem says otherwise explicitly. An answer without work shown will receive little or no credit.
4. If you need more space for your answer, use the back of the page and indicate that you have done so.
5. You may use one sheet of handwritten notes on a 8.5×11 inch paper, both sides. You may use a TI-30X IIS calculator. No other resources are allowed.
6. Raise your hand if you have a question.
7. Time allowed: 110 minutes.

1	/15
2	/15
3	/5
4	/10
5	/15
6	/15
7	/15
8	/10
9	/25
10	/5
Total	/130

(1) (i) [10 points] Compute the determinant of $\begin{bmatrix} x & 1 & 0 \\ y & 2 & 1 \\ z & 0 & 3 \end{bmatrix}$. Show all your work. Your answer should involve x, y, z.

(ii) [5 points] Let P be the plane in $\mathbb{R}^3$ spanned by $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$. Using part (i), find constants a, b, c such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in P \quad \Leftrightarrow \quad ax + by + cz = 0.$$

Explain your answer in a sentence.

(2) Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

(a) [5 points] Find all eigenvalues of A. Show your work, or explain your reasoning.

(b) [8 points] Find a basis for each eigenspace of A. Show all your work, and put a **box around your answer**.

(c) [2 points] Is A diagonalizable? Circle your answer. No justification is needed.

yes no

(3) [5 points] Suppose A is a 3×3 matrix, where applying the following row operations results in an echelon form matrix B:

$$A \sim A_1 \sim A_2 \sim A_3 \sim B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & x \end{bmatrix}.$$

The row operations, in order, are

$$R_1 \leftrightarrow R_2, \qquad R_3 - R_1 \to R_3, \qquad R_3 - R_2 \to R_3, \qquad \frac{1}{2}R_3 \to R_3.$$

What is $\det A$? Show all your work. Your answer should involve the parameter x.

(4) Suppose A is a square matrix such that $\det(A - \lambda I) = \lambda^3(\lambda - 1)^2(\lambda + 1)$.

(a) [2 points] Circle all the eigenvalues of A. No justification is needed.

$$-3 \qquad -2 \qquad -1 \qquad 0 \qquad 1 \qquad 2 \qquad 3$$

(b) [2 points] Circle all possible values for the nullity of A. No justification is needed.

$$0 \qquad 1 \qquad 2 \qquad 3 \qquad 4 \qquad 5 \qquad 6 \qquad 7$$

(c) [6 points] Circle all possible values for the rank of $A^2 - I$.

$$0 \qquad 1 \qquad 2 \qquad 3 \qquad 4 \qquad 5 \qquad 6 \qquad 7$$

Explain your reasoning. (Hint: what are the eigenvalues of A^2?)

(5) Consider the linear transformation

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x + 4z - w \\ y + z + 2w \end{bmatrix}.$$

(a) [3 points] Find a matrix A such that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$. No justification is needed.

(b) [8 points] Find a basis for the kernel of T. Show all your work.

(c) [4 points] Is T one-to-one? Explain your why or why not. You may use parts (a-b) to justify your answer.

(6) (a) [10 points] Let $A = \begin{bmatrix} 0 & x & y \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix}$. Find all values of x and y such that $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ is an eigenvector of A. If no values are possible, write "not possible". Show your work.

(b) [5 points] The matrix A above satisfies

$$\det A = -x^2 - y^2 \qquad \text{and} \qquad A^2 = \begin{bmatrix} x^2 + y^2 & x & y \\ x & x^2 + 1 & xy \\ y & xy & y^2 + 1 \end{bmatrix}$$

Find all (real) values of x and y such that $\det(A^2) = 0$, showing all your work. If no solutions exist, write "not possible".

(Hint: Use properties of the determinant.)

(7) [5 points each] In each of the following, either give an example or write "not possible". Put a **box around your answer**. No justification is necessary.

(a) A matrix A such that the transformation $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ satisfies

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}3\\1\end{bmatrix}, \qquad T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\2\end{bmatrix}, \qquad \text{and} \qquad T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}0\\3\end{bmatrix}.$$

(b) A non-diagonal 2×2 matrix with eigenvalues 3, 4, and 5.

(c) A 3×3 matrix B which has $\det B = 0$ and $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is an eigenvector of B with eigenvalue 8.

(8) Consider the following claim and proof.

Claim: If $\det A = 1$, then the only possible eigenvalues of A are $\lambda = \pm 1$.

Proof: Suppose A is an $n \times n$ matrix.

- $\det(A - \lambda I) = \det(A) - \det(\lambda I) = 1 - \lambda^n$.
- The eigenvalues of A satisfy $\det(A - \lambda I) = 1 - \lambda^n = 0$.
- The only real solutions to $1 - \lambda^n = 0$ are $\lambda = \pm 1$.

(a) [5 points] Explain why the reasoning in the proof is incorrect. (Identify which step in the proof contains mistake. Do not explain by giving a counterexample to the claim.)

(b) [5 points] Find a 2×2 matrix A such that $\det A = 1$ and A does not have 1 or -1 as eigenvalues. No justification is needed.

(9) The matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, which form a basis for $\mathbb{R}^2$.

(a) [5 points] Find matrices P and D such that $A = PDP^{-1}$. (You do not need to calculate P^{-1}.) Show your work, and put a **box around your answer**.

(b) [5 points] Find a matrix C such that

- C is diagonalizable,
- all eigenvalues of C are real, and
- $C^2 = A$,

or explain why this is not possible. Show all your work.
(Hint: how are eigenvectors of C related to eigenvectors of A?)

(c) [5 points] Let $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ be a basis of eigenvectors of A, and let $\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Find $[\vec{\mathbf{x}}]_{\mathcal{B}}$, the coordinates of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in the basis $\mathcal{B}$.

(d) [10 points] Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ as above, which has eigenvectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. (This is not a full list of all eigenvectors of A.) Let

$$S = \{\vec{\mathbf{x}} \in \mathbb{R}^2 : \vec{\mathbf{x}} \text{ is not an eigenvector of } A\}.$$

Is S a subspace of $\mathbb{R}^2$? Justify your answer. Recall that $\vec{\mathbf{0}}$ is not an eigenvector. (Hint: use the definition of a subspace.)

(10) Consider the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, which has determinant $\det A = 2$.

(a) [5 points] The matrix $B = \frac{1}{3}(A + I)$ is the matrix of the averaging transformation on $\mathbb{R}^3$. On your homework, you found that the eigenvalues of B are 0 and 1. Using this observation, what are all the eigenvalues of A? Explain your reasoning. (Hint: What are the eigenvalues of $3B$?)

(b) [Extra credit! 5 points] Suppose A_n is the $n \times n$ matrix with 0's on the main diagonal and 1's off the main diagonal,

$$A_n = \begin{bmatrix} 0 & 1 & 1 & & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & & 1 \\ & \vdots & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}.$$

What is $\det A_n$? (Hint: find the eigenvalues of A_n by thinking of averaging)

END OF EXAM