Math 208 M Spring 2022 Final exam

NAME (First,Last) : ....

Student ID ....

UW email ....

- Please use the same name that appears in Canvas.
- IMPORTANT: Your exam will be scanned: DO NOT write within 1 cm of the edge. Make sure your writing is clear and dark enough.
- Write your NAME (first, last) on top of every ODD page of this exam.
- If you run out of space, continue your work on the back of the last page and indicate clearly on the problem page that you have done so.
- Unless stated otherwise, you **MUST** show your work and justify your answers.
- Please be precise. Imprecise language such as "'this matrix is linearly independent"' or "'this matrix is one to one"' will be marked down.
- Your work needs to be neat and legible.
- This exam contains 4 pages and 6 problems, please make sure you have a complete exam.

Problem 1 Read both parts of this problem, before you start doing calculations. You are given the vectors $v_1 = (1, a, -1)$, $v_2 = (-1, 1, 2)$, $v_3 = (4, -4, a)$.

1. Find all values of a such that v_1, v_2, v_3 do not span R^3.

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ a & 1 & -4 & 2 \\ -1 & 2 & a & b \end{array}\right] \begin{array}{c} r_2 - a r_1 \to r_2 \\ r_3 + r_1 \to r_3 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 0 & 1+a & -4-4a & 2 \\ 0 & 1 & a+4 & b \end{array}\right] \; r_2 \updownarrow r_3 \; \left[\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 0 & 1 & a+4 & b \\ 0 & 1+a & -4(1+a) & 2 \end{array}\right]$$

if $a=-1$ this becomes $\boxed{\left[\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 0 & 1 & 3 & b \\ 0 & 0 & 0 & 2 \end{array}\right] \; a=-1 \; (*)}$, if $a \neq -1$ $\frac{1}{1+a} r_3 \to r_1$ is a legal operation and we get

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 0 & 1 & a+4 & b \\ 0 & 1 & -4 & \frac{2}{a+1} \end{array}\right] r_3 - r_2 \to r_3 \; \boxed{\left[\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 0 & 1 & a+4 & b \\ 0 & 0 & -8-a & \frac{2}{a+1} - b \end{array}\right], a \neq -1}$$

v_1 v_2 v_3 do not span R^3 when $a=-1$ or $a=-8$

2. Give an example of a and b such that the vector $(0, 2, b)$ is not in the span of v_1, v_2, v_3. Show your work to explain how you found your example.

For example if $a=-1$ b can be anything, as we see from (*) above, so $b=0$ would work.

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Problem 2 Suppose that the general solution in vector form of $A\vec{x} = 0$ is $(-x_4, x_4, 2x_4, x_4)$. Answer the following questions; **remember to justify your answers**. If you think that you do not have enough information to answer a question, just answer "'not enough information"'.

1. How many columns does A have ?

Since A can be multiplied by $\begin{pmatrix} -x_4 \\ x_4 \\ 2x_4 \\ x_4 \end{pmatrix}$, then A must have 4 columns

2. How many rows does A have ?

We do not have enough information

3. Find a basis for Null(A), the nullspace of A

All solutions to $Ax=0$ are of the form $x_4(-1,1,2,1)$ so $(-1,1,2,1)$ is a basis for Null (A)

4. What is the rank of A ?

By rank nullity th: rank $= 4 - 1 = 3$

5. Is the first column of A in the span of the other columns of A?

Since for $x_4 = 1$ $\underbrace{\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}}_{A} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = -c_1 + c_2 + 2c_3 + c_4 = 0$

this tells us that $c_1 = c_2 + 2c_3 + c_4$ so yes

Problem 3 Given $W = \{(x,y,z) \text{ in } R^3 : x+y+z=0\}$

Find a 3x3 matrix A such that the null space of A is equal to W or explain why this is not possible.

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ All rows of A need to be of the form (k, k, k) for some k

Find a 3x3 matrix A such that det (A)=1 and the column space of A is equal to W or explain why this is not possible.

W is a plane in R^3 so it has dimension 2, therefore if $Col(A) = W$ rank $A = 2$ so A is not invertible, but then $\det(A) = 0 \neq 1$ IMPOSSIBLE

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Problem 4 Given $A = \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix}$ $\quad A - \lambda I = \begin{bmatrix} 2-\lambda & 5 \\ 5 & 2-\lambda \end{bmatrix}$

1. Find all eigenvalues of A.

$p(\lambda) = (2-\lambda)^2 - 25 = 0 \iff 2-\lambda = \pm 5 \qquad \lambda = 7, -3$

2. For each eigenvalue λ you found, give a basis B for $E(\lambda)$ (the eigenspace of λ).

$E_7 = \text{Null}\begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix}$ basis $v_1 = (1,1)$; $E_{-3} = \text{Null}\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$

basis $v_2 = (1,-1)$

3. Diagonalize A, that is find matrices P and D with $A = PDP^{-1}$.

$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 7 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$

4. The columns of P form a basis B_1 for R^2. Find $[(1,0)]_{B_1}$, the vector of coordinates of (1,0) with respect to B_1.

you can either calculate P^{-1} and then $P^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = [(1,0)]_{B_1}$

or solve $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 0 \end{array}\right] r_2 - r_1 \to r_2 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix} \quad x_2 = \frac{1}{2},\ x_1 = 1 - x_2 = \frac{1}{2}$

$[(1,0)]_{B_1} = (\frac{1}{2}, \frac{1}{2})$ meaning that $(1,0) = \frac{1}{2}(1,1) + \frac{1}{2}(1,-1)$

5. Find $[A\begin{pmatrix} 1 \\ 0 \end{pmatrix}]_{B_1}$ (Hint: can you use D ?)

$D[\begin{pmatrix} 1 \\ 0 \end{pmatrix}]_{B_1} = [A\begin{pmatrix} 1 \\ 0 \end{pmatrix}]_{B_1}$

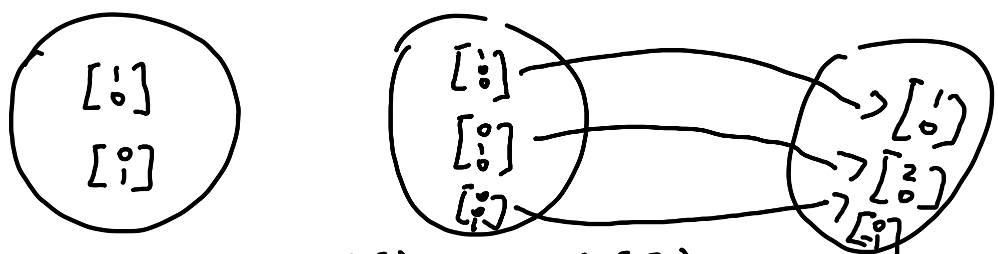
$D[\begin{pmatrix} 1 \\ 0 \end{pmatrix}]_{B_1} = \begin{bmatrix} 7 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 7/2 \\ -3/2 \end{bmatrix}$

Problem 5 Consider the linear transformation $T : R^3 \to R^2$ defined by $T(v) = \underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{M} v$

1. is T one to one ? Justify your answer.

No $3 > 2$, or: the columns of M are not linearly independent

2. Is it possible to find a linear transformation $S : R^2 \to R^3$ $S(v) = Bv$ for some matrix B, such that $T \circ S$, the composition of S and T, (recall that this means that $T \circ S(v) = T(S(v))$) is one to one ? If you think it is possible, find B, and show your work to explain how you found B. If you think this is not possible, justify why.



$T \circ S$ is one to one $\iff$ $T(S\begin{bmatrix}1\\0\end{bmatrix})$ and $T(S\begin{bmatrix}0\\1\end{bmatrix})$ are linearly independent. I can see that $T(\begin{bmatrix}1\\0\\0\end{bmatrix})$ and $T(\begin{bmatrix}0\\0\\1\end{bmatrix})$ are linearly independent so setting $S\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $S\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$, that is $B = \begin{pmatrix}1 & 0\\0 & 0\\0 & 1\end{pmatrix}$ would work.

OR $B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ $\quad AB = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ -e & -f \end{bmatrix} = \beta$

We need $\mathrm{Null}(\beta) = \{\vec{0}\}$ so, by the unifying theorem, $\det \beta \neq 0$

pick any values for a, b, c, d, e, f such that

$-f(a+2c) + e(b+2d) \neq 0$

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Problem 6 Suppose A is a 3×3 matrix with characteristic polynomial $p(\lambda) = -\lambda(\lambda - 2)^2$ and that the eigenspace for eigenvalue $\lambda = 2$ is $E_2 = \{(x, y, z) \text{ in } R^3 : x + y + z = 0\}$

1. Is A invertible ? Justify your answer.

No $\lambda = 0$ is an eigenvalue for A

2. Is A diagonalizable ? Justify your answer. yes

$AM(0)=1$ and $GM(0) = \dim E_0$ must then be between 1 and 1
so $GM(0)=1$ and $E_0 = \text{span}(v)$ for some v
E_2 is a plane in R^3 so it has dimension 2, therefore
$E_2 = \text{span}(v_1\ v_2)$ and $v\ v_1\ v_2$ is linearly independent,
because when we put together bases of eigenspaces we
always get linearly independent vectors. So $v\ v_1 v_2$
is a basis for R^3
consisting of eigenvectors
of A. We can diagonalize
A. $A = \begin{bmatrix} v & v_1 & v_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v & v_1 & v_2 \end{bmatrix}^{-1}$

3. What is the rank of A ? Justify your answer.

By the discussion above nullity $A = 1$
so rank $= 3 - 1 = 2$