Your Name

Your Signature

Student ID #

- **Use the backs of the pages** when you have run out of room. Both sides of every page will be scanned, and it is expected for some questions that you may need to use the back of the page.
- Turn off all cell phones, pagers, radios, mp3 players, and other similar devices.
- This exam is closed book. You may use one $8.5'' \times 11''$ sheet of handwritten notes (both sides OK). Do not share notes. No photocopied materials are allowed.
- Graphing calculators are not allowed.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Raise your hand if you have a question.
- This exam has 10 pages, plus this cover sheet. Please make sure that your exam is complete.

Question	Points	Score
1	11	
2	20	
3	14	
4	13	
5	14	
6	8	
Total	80	

1. (11 points) **For this problem, you do not have to show work or justification.** For each of the following statements, circle "T" to the left if the statement is true, and "F" if the statement is false. **Here "true" means "always true".** If the are both examples of and counterexamples to the statement, the correct answer is "false." If you don't know the answer almost immediately, just make a guess and move on; time is better spent on the other exam questions.

$\boxed{T}$	F	If $T(\mathbf{x}) = A\mathbf{x}$ and $S(\mathbf{x}) = B\mathbf{x}$, then $T \circ S(\mathbf{x}) = AB\mathbf{x}$.
T	$\boxed{F}$	If B is the reduced echelon form for A, then $\det B = \det A$.
T	$\boxed{F}$	If $T : \mathbb{R}^3 \to \mathbb{R}^5$ is a linear transformation, then $\dim \operatorname{range}(T) \geq 3$.
$\boxed{T}$	F	If $\mathbf{v_1}, \ldots, \mathbf{v_m}$ is a linearly dependent list, then one of the vectors is in the span of the rest.
$\boxed{T}$	F	If $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ spans $\mathbb{R}^2$, then $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{w}$ also spans $\mathbb{R}^2$.
T	$\boxed{F}$	If A and B are 2×2 matrices, then $AB = BA$.
T	$\boxed{F}$	If $T : \mathbb{R}^5 \to \mathbb{R}^2$ is a linear transformation and $\ker T \neq \{\mathbf{0}\}$, then T is not surjective (onto).
$\boxed{T}$	F	If $\mathbf{v_1}, \mathbf{v_2}$ span $\mathbb{R}^2$, then $\mathbf{v_1}, \mathbf{v_2}$ is a linearly independent list.
$\boxed{T}$	F	If a linear system has more than one solution, it has infinitely many solutions.
$\boxed{T}$	F	$\mathbb{R}^3$ contains no list of 4 linearly independent vectors.
$\boxed{T}$	F	If the columns of the matrix A span $\mathbb{R}^4$, then the associated linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is surjective (onto).

2. (20 points) (This problem is continued on the next page.) For each of the following, give an example of the object described, **or** briefly explain why such an example cannot exist. (You can make your examples as simple as you want, as long as they satisfy the properties.) You do not have to show that your example satisfies the properties (although you may check if you are unsure).

   (a) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which has no eigenvectors (in $\mathbb{R}^2$). [You can give the transformation either by giving its matrix or giving a geometric description, as long as the description is detailed enough to be unambiguous.]
   Solution. e.g. T is rotation about the origin by 90 degrees.

   (b) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which is injective (one-to-one) but not surjective (onto).
   Solution. This is impossible by the Unifying Theorem.

   (c) A list of four different nonzero vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4} \in \mathbb{R}^4$ with $\dim \operatorname{span}(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}) = 2$.
   Solution. e.g. $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

   (d) A linearly independent list of 3 vectors in $\mathbb{R}^3$ which all solve the equation $x_1 - 2x_2 + x_3 = 0$.
   Solution. Impossible; the solution set to one homogeneous equation in three variables is a 2 dimensional subspace, so it cannot contain an independent list of 3 vectors.

   (e) A square matrix with rank 2 and nullity 2.
   Solution. e.g. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

(Problem 2 continued)

(f) A system of 2 linear equations in 3 variables which has no solution.

Solution. e.g. the system represented by the augmented matrix $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$.

(g) A system of 2 linear equations in 3 variables that has a unique solution.

Solution. Impossible; a system with more variables than equations must have at least one free variable, so it must have either 0 or infinitely many solutions.

(h) A 2×2 invertible matrix A with $\det(A^{-1}) = 0$.

Solution. Impossible; if A is invertible, then so is A^{-1}, and invertible matrices have nonzero determinant.

(i) A list of 3 vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ which span $\mathbb{R}^3$ and such that $2\mathbf{v_1}, -\mathbf{v_2}, \mathbf{v_3}$ *does not* span $\mathbb{R}^3$.

Solution. Impossible; multiplying a list of vectors by nonzero scalars does not change its span.

(j) A 2×2 matrix A with all nonzero entries such that $T(\mathbf{x}) = A\mathbf{x}$ is not surjective (onto).

Solution. e.g. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

3. (14 points) Consider the matrix
$$A = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$
The characteristic polynomial of A is $\lambda^2(1-\lambda)(1+\lambda)$.

(a) What are the eigenvalues of A?
Solution. These are simply the roots of the characteristic polynomial, so 0 (double root), $1, -1$.

(b) What are the eigenvalues corresponding to the eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ respectively?

Solution. We can compute
$$A\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad A\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = (-1)\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix},$$
so the first eigenvector has eigenvalue 1, and the second eigenvector has eigenvalue -1.

(c) Compute a basis for $\mathbb{R}^4$ consisting of eigenvectors for A.
Solution. In part (b) we were already given bases for the eigenspaces E_1 and E_{-1}, so now we compute a basis for $E_0 = \text{null}(A)$, using row operations to find the parametric form of the solution set to $A\mathbf{x} = \mathbf{0}$:
$$\text{null}(A) = \text{null}\left(\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right).$$
Combining the bases for all the eigenspaces together gives a basis for $\mathbb{R}^4$ consisting of eigenvectors for A: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

(d) Give a diagonalization for A. (You don't have to compute P^{-1}; just give an invertible P and a diagonal D such that $A = PDP^{-1}$.)
Solution. $A = PDP^{-1}$, where
$$P = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(e) Compute A^9.
Solution. From part (d) we know that (using the same P) a diagonalization for A^9 is given by
$$A^9 = P\begin{bmatrix} 1^9 & 0 & 0 & 0 \\ 0 & (-1)^9 & 0 & 0 \\ 0 & 0 & 0^9 & 0 \\ 0 & 0 & 0 & 0^9 \end{bmatrix}P^{-1} = P\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}P^{-1},$$

which is our diagonalization for A. So

$$A^9 = A = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

(f) Compute a diagonalization for A^2, and describe A^2 geometrically [e.g. if A^2 is a reflection through or projection onto a subspace, be specific about what subspace it is].
Solution. From part (d) we know that $A^2 = PD'P^{-1}$ where

$$P = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \quad D' = D^2 = \begin{bmatrix} (1)^2 & 0 & 0 & 0 \\ 0 & (-1)^2 & 0 & 0 \\ 0 & 0 & 0^2 & 0 \\ 0 & 0 & 0 & 0^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the eigenvalues of A^2 are just 0 and 1, A^2 is a projection onto the plane given by its 1-eigenspace

$$E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right).$$

4. (13 points) (This problem continues on the next page.) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 3 & 0 & -1 & -1 \\ 0 & 0 & -1 & 2 \\ 2 & 1 & -1 & 2 \end{bmatrix}.$$

The reduced echelon form of A is

$$B = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Write the last (rightmost) column of A as a linear combination of the first three columns. [Give explicit coefficients.]

Solution.

$$\begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + (-2)\begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}.$$

(b) Is there more than one correct choice of coefficients for (a)? Explain how you know.

Solution. No; from looking at the reduced echelon form of A we can see that its first 3 columns form an independent list. Therefore, the solution to the system $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 3 & 0 & -1 & -1 \\ 0 & 0 & -1 & 2 \\ 2 & 1 & -1 & 2 \end{array}\right]$ is unique.

(c) How many solutions are there to the system

$$\left[\begin{array}{c|c} & 0 \\ A & 0 \\ & 0 \\ & 1 \end{array}\right]?$$

Briefly explain how you know.

Solution. Infinitely many solutions. The system is consistent, since clearly $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in \operatorname{col}(A)$. Looking at the reduced echelon form of A we see that the fourth column does not contain a pivot (is a free variable); so there are infinitely many solutions.

(d) How many solutions are there to the system

$$\left[\begin{array}{c|c} & 0 \\ B & 0 \\ & 0 \\ & 1 \end{array}\right]?$$

Briefly explain how you know.

Solution. There are no solutions, since the bottom row of the system $[0000|1]$ represents a contradictory equation.

(Problem 4 continued; recall that

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 3 & 0 & -1 & -1 \\ 0 & 0 & -1 & 2 \\ 2 & 1 & -1 & 2 \end{bmatrix}, \quad B = REF(A) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.)$$

(e) Give a basis for $\text{row}(A)$.

Solution. e.g. $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$. [Note that you also could have found an independent sublist of the rows of A of size 3, if you did more work. Note that the first 3 rows of A do *not* form a basis for $\text{row}(A)$ in this case.]

(f) Give a basis for $\text{col}(A)$.

Solution. e.g. $\begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$. [In this case, any 3 of the columns of A would also work].

(g) What is the rank and nullity of A?

Solution. $\text{rank}(A) = 3$, $\text{nullity}(A) = 1$.

(h) Does A have an inverse?

Solution. No (since its reduced echelon form has fewer pivots than it has rows/columns).

5. (14 points) Consider the matrices

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) For which values of λ does the equation

$$(A - \lambda B)\mathbf{x} = \mathbf{0}$$

have a nonzero solution?

Solution. By the Unifying Theorem, that equation has a nonzero solution exactly when $\det(A - \lambda B) = 0$. So we compute (using cofactor expansion along the bottom row)

$$\det(A - \lambda B) = \det \begin{bmatrix} 2 & -\lambda & 0 \\ -\lambda & 8 & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)\det \begin{bmatrix} 2 & -\lambda \\ -\lambda & 8 \end{bmatrix}$$
$$= (1-\lambda)(16-\lambda^2) = (1-\lambda)(4-\lambda)(4+\lambda),$$

which is zero exactly when λ is $1, 4$, or -4.

(b) Find a nonzero $\mathbf{v} \in \mathbb{R}^3$ such that $A\mathbf{v}$ and $B\mathbf{v}$ point in the same direction, but $A\mathbf{v} \neq B\mathbf{v}$. [Hint: rearrange and try to interpret the equation in (a).]

Solution. The equation from (a) is equivalent to the equation $A\mathbf{x} = \lambda B\mathbf{x}$. So $A\mathbf{v}$ and $B\mathbf{v}$ point in the same direction exactly when $\mathbf{v}$ solves the equation from (a) for some $\lambda > 0$. We have to exclude the case $\lambda = 1$, since in that case we would have $A\mathbf{v} = B\mathbf{v}$. By part (a), the only value of λ left where the equation has a nonzero solution is $\lambda = 4$. So we want $\mathbf{v}$ to be a nonzero element of

$$\operatorname{null}(A - 4B) = \begin{bmatrix} 2 & -4 & 0 \\ -4 & 8 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \operatorname{null} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

so we can take, for example, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

(c) What are the eigenvalues of $B^{-1}A$? [Hint: you may be able to save some computation by relating the eigenvector equation for $B^{-1}A$ to the equation from (a).]

Solution. λ is an eigenvalue of $B^{-1}A$ exactly when it is a root of the characteristic polynomial, i.e.

$$\det(B^{-1}A - \lambda I_3) = \det(B^{-1}(A - \lambda B)) = \det(B^{-1})\det(A - \lambda B) = 0$$

(here we have used the distributive property and linearity of matrix multiplication, as well as multiplicativity of the determinant). Since B is invertible, so $\det(B^{-1}) \neq 0$, the above equation holds exactly $\det(A - \lambda B) = 0$. So by our computation from (a), the eigenvalues of $B^{-1}A$ are $1, 4, -4$.

(d) What are the eigenvalues of $A^{-1}B$? [Hint: how does $A^{-1}B$ relate to $B^{-1}A$?]

Solution. Since $A^{-1}B = (B^{-1}A)^{-1}$, we know that its eigenvalues are the inverses of the eigenvalues of $B^{-1}A$, that is, $1, \frac{1}{4}, -\frac{1}{4}$.

(e) What is $\det(A^{-1}BA^{-1}BA^{-1}B)$? (You can leave exponents in your answer instead of multiplying it out.)

Solution. By multiplicativity of the determinant, $\det(A^{-1}BA^{-1}BA^{-1}B) = \det(A^{-1}B)^3$, and since the determinant of a matrix is the product of its eigenvalues, this is equal to $[1 \cdot \frac{1}{4} \cdot (-\frac{1}{4})]^3 = -\frac{1}{16^3}$. [One could also quickly calculate that $\det B = -1$ and $\det A = 16$ and get the same answer.]

6. (8 points) Consider the planes

$$P_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right), \quad P_2 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}\right).$$

(a) Find a nontrivial linear equation, i.e. an equation of the form

$$ax_1 + bx_2 + cx_3 = 0$$

which is not $0x_1 + 0x_2 + 0x_3 = 0$, which is satisfied by every $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in P_1$.

Solution. Any such equation which is satisfied by a list is also satisfied by any vector in the span of that list. So an equation that works is $x_3 = 0$.

(b) Using the Rank-Nullity theorem, argue that if we find another equation

$$dx_1 + ex_2 + fx_3 = 0$$

which is satisfied by all $\mathbf{x} \in P_1$, then it must be a multiple of your answer from (a); that is, for some $t \in \mathbb{R}$, we must have $\begin{bmatrix} d \\ e \\ f \end{bmatrix} = t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. [Hint: First notice that $P_1 \subseteq \operatorname{null} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$.]

Solution. First, note that if both equations hold for all $\mathbf{x} \in P_1$, this means that $P_1 \subseteq \operatorname{null} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$. Since P_1 is two dimensional (the two vectors in the list are clearly linearly independent, since neither is a multiple of the other), this then means that

$$\operatorname{nullity}\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \dim \operatorname{null} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \geq 2.$$

Therefore, the Rank-Nullity theorem tells us that

$$\dim \operatorname{row}\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) \leq 1.$$

But if $\begin{bmatrix} d \\ e \\ f \end{bmatrix}$ were not a multiple of $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, then the row space would have dimension 2. Therefore $\begin{bmatrix} d \\ e \\ f \end{bmatrix}$ is a multiple of $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

(c) As in (a), find a nontrivial linear equation satisfied by every $\mathbf{x} \in P_2$.

Solution. $x_2 = 2x_1$, or equivalently, $2x_1 - x_2 = 0$.

(d) Write a system of linear equations whose solution set is the intersection of P_1 and P_2 (i.e. all the points which lie in both P_1 and P_2).

Solution. Combining our answers from (a) and (c), the intersection of P_1 and P_2 is the solution set to the system

$$\begin{aligned} 0x_1 + 0x_2 + 1x_3 &= 0 \\ 2x_1 + 1x_2 + 0x_3 &= 0. \end{aligned}$$

(e) Write the intersection of P_1 and P_2 as the span of a list of vectors.

Solution. It is easy to see that the system in (d) has two pivots and one free variable [e.g. switching rows puts it in echelon form], so the solution set is one dimensional; hence we can write it as the span of any nonzero vector which solves both equations. Therefore, the intersection of P_1 and P_2 is span $\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right)$.