Your Name

Your Signature

Student ID #

- **Use the backs of the pages** when you have run out of room. Both sides of every page will be scanned, and it is expected for some questions that you may need to use the back of the page.
- Turn off all cell phones, pagers, radios, mp3 players, and other similar devices.
- This exam is closed book. You may use one $8.5'' \times 11''$ sheet of handwritten notes (both sides OK). Do not share notes. No photocopied materials are allowed.
- Graphing calculators are not allowed.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Raise your hand if you have a question.
- This exam has 8 pages, plus this cover sheet. Please make sure that your exam is complete.

Question	Points	Score
1	11	
2	20	
3	14	
4	13	
5	14	
6	8	
Total	80	

1. (11 points) **For this problem, you do not have to show work or justification.** For each of the following statements, circle "T" to the left if the statement is true, and "F" if the statement is false. **Here "true" means "always true".** If the are both examples of and counterexamples to the statement, the correct answer is "false." If you don't know the answer almost immediately, just make a guess and move on; time is better spent on the other exam questions.

T	F	If $T(\mathbf{x}) = A\mathbf{x}$ and $S(\mathbf{x}) = B\mathbf{x}$, then $T \circ S(\mathbf{x}) = AB\mathbf{x}$.
T	F	If B is the reduced echelon form for A, then $\det B = \det A$.
T	F	If $T : \mathbb{R}^3 \to \mathbb{R}^5$ is a linear transformation, then $\dim \operatorname{range}(T) \geq 3$.
T	F	If $\mathbf{v_1}, \ldots, \mathbf{v_m}$ is a linearly dependent list, then one of the vectors is in the span of the rest.
T	F	If $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ spans $\mathbb{R}^2$, then $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{w}$ also spans $\mathbb{R}^2$.
T	F	If A and B are 2×2 matrices, then $AB = BA$.
T	F	If $T : \mathbb{R}^5 \to \mathbb{R}^2$ is a linear transformation and $\ker T \neq \{\mathbf{0}\}$, then T is not surjective (onto).
T	F	If $\mathbf{v_1}, \mathbf{v_2}$ span $\mathbb{R}^2$, then $\mathbf{v_1}, \mathbf{v_2}$ is a linearly independent list.
T	F	If a linear system has more than one solution, it has infinitely many solutions.
T	F	$\mathbb{R}^3$ contains no list of 4 linearly independent vectors.
T	F	If the columns of the matrix A span $\mathbb{R}^4$, then the associated linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is surjective (onto).

2. (20 points) (This problem is continued on the next page.) For each of the following, give an example of the object described, **or** briefly explain why such an example cannot exist. (You can make your examples as simple as you want, as long as they satisfy the properties.) You do not have to show that your example satisfies the properties (although you may check if you are unsure).

   (a) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which has no eigenvectors (in $\mathbb{R}^2$). [You can give the transformation either by giving its matrix or giving a geometric description, as long as the description is detailed enough to be unambiguous.]

   (b) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which is injective (one-to-one) but not surjective (onto).

   (c) A list of four different nonzero vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4} \in \mathbb{R}^4$ with $\dim \operatorname{span}(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}) = 2$.

   (d) A linearly independent list of 3 vectors in $\mathbb{R}^3$ which all solve the equation $x_1 - 2x_2 + x_3 = 0$.

   (e) A square matrix with rank 2 and nullity 2.

(Problem 2 continued)

(f) A system of 2 linear equations in 3 variables which has no solution.

(g) A system of 2 linear equations in 3 variables that has a unique solution.

(h) A 2×2 invertible matrix A with $\det(A^{-1}) = 0$.

(i) A list of 3 vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ which span $\mathbb{R}^3$ and such that $2\mathbf{v_1}, -\mathbf{v_2}, \mathbf{v_3}$ *does not* span $\mathbb{R}^3$.

(j) A 2×2 matrix A with all nonzero entries such that $T(\mathbf{x}) = A\mathbf{x}$ is not surjective (onto).

3. (14 points) Consider the matrix

$$A = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of A is $\lambda^2(1-\lambda)(1+\lambda)$.

(a) What are the eigenvalues of A?

(b) What are the eigenvalues corresponding to the eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ respectively?

(c) Compute a basis for $\mathbb{R}^4$ consisting of eigenvectors for A.

(d) Give a diagonalization for A. (You don't have to compute P^{-1}; just give an invertible P and a diagonal D such that $A = PDP^{-1}$.)

(e) Compute A^9.

(f) Compute a diagonalization for A^2, and describe A^2 geometrically [e.g. if A^2 is a reflection through or projection onto a subspace, be specific about what subspace it is].

4. (13 points) (This problem continues on the next page.) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 3 & 0 & -1 & -1 \\ 0 & 0 & -1 & 2 \\ 2 & 1 & -1 & 2 \end{bmatrix}.$$

The reduced echelon form of A is

$$B = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Write the last (rightmost) column of A as a linear combination of the first three columns. [Give explicit coefficients.]

(b) Is there more than one correct choice of coefficients for (a)? Explain how you know.

(c) How many solutions are there to the system

$$\left[\begin{array}{c|c} & 0 \\ A & 0 \\ & 0 \\ & 1 \end{array}\right]?$$

Briefly explain how you know.

(d) How many solutions are there to the system

$$\left[\begin{array}{c|c} & 0 \\ B & 0 \\ & 0 \\ & 1 \end{array}\right]?$$

Briefly explain how you know.

(Problem 4 continued; recall that

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 3 & 0 & -1 & -1 \\ 0 & 0 & -1 & 2 \\ 2 & 1 & -1 & 2 \end{bmatrix}, \quad B = REF(A) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.)$$

(e) Give a basis for $\text{row}(A)$.

(f) Give a basis for $\text{col}(A)$.

(g) What is the rank and nullity of A?

(h) Does A have an inverse?

5. (14 points) Consider the matrices

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   (a) For which values of λ does the equation

$$(A - \lambda B)\mathbf{x} = \mathbf{0}$$

   have a nonzero solution?

   (b) Find a nonzero $\mathbf{v} \in \mathbb{R}^3$ such that $A\mathbf{v}$ and $B\mathbf{v}$ point in the same direction, but $A\mathbf{v} \neq B\mathbf{v}$. [Hint: rearrange and try to interpret the equation in (a).]

   (c) What are the eigenvalues of $B^{-1}A$? [Hint: you may be able to save some computation by relating the eigenvector equation for $B^{-1}A$ to the equation from (a).]

   (d) What are the eigenvalues of $A^{-1}B$? [Hint: how does $A^{-1}B$ relate to $B^{-1}A$?]

   (e) What is $\det(A^{-1}BA^{-1}BA^{-1}B)$? (You can leave exponents in your answer instead of multiplying it out.)

6. (8 points) Consider the planes

$$P_1 = \operatorname{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right), \quad P_2 = \operatorname{span}\left(\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\4\\1\end{bmatrix}\right).$$

   (a) Find a nontrivial linear equation, i.e. an equation of the form

$$ax_1 + bx_2 + cx_3 = 0$$

   which is not $0x_1 + 0x_2 + 0x_3 = 0$, which is satisfied by every $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in P_1$.

   (b) Using the Rank-Nullity theorem, argue that if we find another equation

$$dx_1 + ex_2 + fx_3 = 0$$

   which is satisfied by all $\mathbf{x} \in P_1$, then it must be a multiple of your answer from (a); that is, for some $t \in \mathbb{R}$, we must have $\begin{bmatrix}d\\e\\f\end{bmatrix} = t\begin{bmatrix}a\\b\\c\end{bmatrix}$. [Hint: First notice that $P_1 \subseteq \operatorname{null}\begin{bmatrix}a & b & c\\ d & e & f\end{bmatrix}$.]

   (c) As in (a), find a nontrivial linear equation satisfied by every $\mathbf{x} \in P_2$.

   (d) Write a system of linear equations whose solution set is the intersection of P_1 and P_2 (i.e. all the points which lie in both P_1 and P_2).

   (e) Write the intersection of P_1 and P_2 as the span of a list of vectors.