308 Final Exam Autumn 2018

Math 308R, Final Exam Name: ____________________

Signature: ____________________________________

Student ID #: ______________ Section #: ______________

- You are allowed a Ti-30x IIS Calculator and one 8.5×11 inch paper with notes on both sides. Other calculators, electronic devices (e.g. cell phones, laptops, etc.), notes, and books are **not** allowed.
- *All* answers on the exam must be justified. You will receive at most 1 point out of 10 for an answer without any explanation.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Raise your hand if you have a question.
- None of the questions require long and involved calculations. If you find yourself doing that, then pause, take a step back, and think if there is another way you can solve the problem.

1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
9	/10
10	/10
11	/10
12	/10
T	/120

Good Luck!

(1) [10pts] Determine the solution set to the matrix equation

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 2 & 1 & -2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 1 & 2 & 1 & -2 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 2 & 0 & -2 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Free variables: $x_3 = s$, $x_4 = t$.

Thus, $x_2 = 1+t$ $x_1 = -s$

$$\underline{x} = \begin{bmatrix} -s \\ 1+t \\ s \\ t \end{bmatrix}$$

The solution set is therefore:

$$\boxed{\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad s,t \in \mathbb{R} \right\}}$$

(2) [10pts] Determine whether the following vectors span $\mathbf{R}^4$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix} \right\}$$

In order for vectors to span $\mathbb{R}^4$ we must have 4 linearly independent vectors. Among the above 4 vectors, one is the zero vector, making them 4 vectors that are not linearly independent. Therefore, the vectors do not span $\mathbb{R}^4$.

(3) [10pts] Determine whether the following set of vectors in $\mathbf{R}^4$ are linearly independent.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$$

In $\mathbb{R}^4$, 5 or more vectors always are linearly dependent.
The above set has 5 vectors in $\mathbb{R}^4$, so they are
<u>not</u> linearly independent.

(4) [10pts] Let $T\colon \mathbf{R}^4 \to \mathbf{R}^3$ be the linear transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_4 \\ x_2 + x_3 \\ x_1 - x_2 - x_3 + x_4 \end{bmatrix}.$$

Is T onto?

The corresponding matrix is $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{bmatrix}$. This matrix is equivalent with $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which has rank 2, so it does not map onto $\mathbb{R}^3$.

(5) Find an example of a linear transformation $T_1 \colon \mathbf{R}^3 \to \mathbf{R}^2$ and a linear transformation $T_2 \colon \mathbf{R}^2 \to \mathbf{R}^4$ so that the composition

$$T(\mathbf{x}) = T_2(T_1(\mathbf{x})) = (T_2 \circ T_1)(\mathbf{x})$$

is one-to-one, OR explain why this is impossible. If you provide an example, your answer should include an explanation of why this linear transformation has the desired properties.

As linear transformations $\mathbb{R}^n \to \mathbb{R}^m$ with $n > m$ (here $3 > 2$) are not one-to-one, the composition is also not one-to-one.

More detailed:

T_1 is not one-to-one, so there are $\underline{v}_1 \neq \underline{v}_2 \in \mathbb{R}^3$ with $T_1(\underline{v}_1) = T_1(\underline{v}_2)$. But then $T_2 \circ T_1(\underline{v}_1) = T_2 \circ T_1(\underline{v}_2)$, so not one-to-one.

(6) Find an example of
- a linear transformation $T \colon \mathbf{R}^3 \to \mathbf{R}^4$, **and**
- linearly **dependent** vectors $\mathbf{u}$ and $\mathbf{v}$
- such that $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly **independent**,

OR explain why this is impossible. If you provide an example, your answer should include an explanation of why this linear transformation and these vectors have the desired properties.

This is <u>not</u> possible.

Let $\underline{u}, \underline{v} \in \mathbb{R}$ be lin dependent. Then there is $a \in \mathbb{R}$ with $\underline{u} = a \cdot \underline{v}$ or $\underline{v} = a \cdot \underline{u}$ (if $\underline{0} = \underline{v}$, $\underline{u} \neq \underline{0}$ first equation does not hold so we must use the second).

If T is a linear transformation we then have

$T(\underline{u}) = T(a\underline{v}) = a\,T(\underline{v})$ or $T(\underline{v}) = T(a\underline{u}) = a\,T(\underline{u})$

lin. transf. property lin trans. property.

meaning $T(\underline{u})$ and $T(\underline{v})$ always are linearly dependent.

(7) Let A and B be the following equivalent 4×5 matrices.

$$A = \begin{bmatrix} 1 & 0 & 3 & 1 & 1 \\ 2 & 0 & 3 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for col(A) that includes $\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$.

We see that rank $B = 2$, so $\dim(\mathrm{col}(A)) = \dim(\mathrm{col}(B)) = \mathrm{rank}\, B = 2$

A basis for col(A) could be

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}}$$

(8) Let A be a 3×3 matrix. We perform the following row operations and get to the matrix B below. (A_1 and A_2 denote the intermediate matrices between the row operations.)

$$A \overset{R_1 \leftrightarrow R_3}{\rightsquigarrow} A_1 \overset{R_2 = R_2 + 3R_1}{\rightsquigarrow} A_2 \overset{R_3 = R_3 - R_1}{\rightsquigarrow} B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute $\det(A)$.

Switching two rows results in a negative sign in the determinant. $\rightarrow \det(A) = -\det(A_1)$

Adding multiples of one row to another does not change the determinant

$\rightarrow \det A = -\det(A_1) = -\det(A_2) = -\det(B)$.

$\det B = 2 \cdot 3 \cdot 1 = 6$, so $\boxed{\det A = -6}$

(9) [10pts] Determine the eigenvalues and dimensions of the eigenspaces of

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$$\chi_A = \begin{vmatrix} \lambda-2 & 0 & 0 & 0 \\ 0 & \lambda-2 & -1 & -1 \\ 0 & 0 & \lambda-4 & \\ 0 & 0 & -1 & \lambda \end{vmatrix} = (\lambda-2)\begin{vmatrix} \lambda-2 & -1 & -1 \\ 0 & \lambda & -4 \\ 0 & -1 & \lambda \end{vmatrix} = (\lambda-2)^2 \begin{vmatrix} \lambda & -4 \\ -1 & \lambda \end{vmatrix} = (\lambda-2)^2(\lambda^2-4)$$

$$= (\lambda-2)^3(\lambda+2)$$

So eigenvalues are when $\chi_A(\lambda) = 0$ i.e. $\lambda = 2$ or $\lambda = -2$.

$\lambda = 2$: To find the Eigenspace is to find the nullspace of

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The nullspace is

$$\boxed{E_2 = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle \quad \dim E_2 = 2}$$

$\lambda = -2$: Nullspace of

$$\begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -4 & -1 & -1 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & -1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/4 & 1/4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is

$$\boxed{E_{-2} = \left\langle \begin{bmatrix} 0 \\ 1/4 \\ -2 \\ 1 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 0 \\ 1 \\ -8 \\ 4 \end{bmatrix} \right\rangle \quad \dim E_{-2} = 1}$$

(10) [10pts] Let A be the 2×2 matrix

$$A = \begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix}.$$

$$\begin{bmatrix} 1-\lambda & 3 \\ 5 & -1-\lambda \end{bmatrix}$$

The characteristic polynomial of A is $(4-\lambda)(4+\lambda)$. Compute an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

Eigenvalues : $\lambda = 4$, $\lambda = -4$

Eigenspaces: E_4 : $\begin{bmatrix} -3 & 3 \\ 5 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad E_4 = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$

E_{-4} $\begin{bmatrix} 5 & 3 \\ 5 & 3 \end{bmatrix} \to \begin{bmatrix} 5 & 3 \\ 0 & 0 \end{bmatrix}$ $\quad E_{-\lambda} = \left\langle \begin{bmatrix} -3/5 \\ 1 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\rangle$

Thus,

$$\boxed{P = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}}$$

(11) Let A be an invertible 3×3 matrix and let B be a 3×4 matrix with nullity 2. What are the possible values for the nullity of AB?

$A \cdot B$ is a 3×4 matrix.

Claim:

nullity (AB) = nullity $(B) = 2$.

If $\underline{w} \in \mathbb{R}^4$ with

$(A \cdot B)\underline{w} = \underline{0}$, then

$A \cdot (B\underline{w}) = \underline{0} \iff B\underline{w} = A^{-1} \cdot \underline{0} \iff B\underline{w} = \underline{0}$

So $\underline{w} \in$ nullspace (AB) if and only if $\underline{w} \in$ nullspace (B).

So nullspace (AB) = nullspace (B), so

nullity (AB) = nullity (B).

(12) [10pts] Let A be a 2×2 matrix with 0 and 1 as eigenvalues and let $B = A^2 + A$. Determine the nullity of B. (*Hint:* $B = (A + I)A = A(A + I)$.)

If A has Eigenvalues $0, 1$, then its nullity > 0 (0 is EV)

and its nullity < 2 (it is not 0 matrix as 1 is EV)

So nullity $A = 1$.

Then nullity $(B) = \text{nullity}((A+I)A) \geq \text{nullity}(A) = 1$

Also note that if $0 \neq \underline{v} \in E_1$, then

$$B \cdot \underline{v} = (A+I)\underbrace{A \cdot \underline{v}}_{1 \cdot \underline{v}} = (A+I)\underline{v} = A \cdot \underline{v} + \underline{v} = \underline{v} + \underline{v} = 2\underline{v} \neq 0$$

so $B \neq 0$-matrix which means nullity $B \geq 1$

Thus $\boxed{\text{nullity } (B) = 1}$