Math 308R, Final Exam Name: ______________________

Signature: ______________________________________

Student ID #: ______________ Section #: ______________

- You are allowed a Ti-30x IIS Calculator and one 8.5×11 inch paper with notes on both sides. Other calculators, electronic devices (e.g. cell phones, laptops, etc.), notes, and books are **not** allowed.
- *All* answers on the exam must be justified. You will receive at most 1 point out of 10 for an answer without any explanation.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Raise your hand if you have a question.
- None of the questions require long and involved calculations. If you find yourself doing that, then pause, take a step back, and think if there is another way you can solve the problem.

1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
9	/10
10	/10
11	/10
12	/10
T	/120

Good Luck!

(1) [10pts] Determine the solution set to the matrix equation

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 2 & 1 & -2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

(2) [10pts] Determine whether the following vectors span $\mathbf{R}^4$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix} \right\}$$

(3) [10pts] Determine whether the following set of vectors in $\mathbf{R}^4$ are linearly independent.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$$

(4) [10pts] Let $T\colon \mathbf{R}^4 \to \mathbf{R}^3$ be the linear transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_4 \\ x_2 + x_3 \\ x_1 - x_2 - x_3 + x_4 \end{bmatrix}.$$

Is T onto?

(5) Find an example of a linear transformation $T_1 \colon \mathbf{R}^3 \to \mathbf{R}^2$ and a linear transformation $T_2 \colon \mathbf{R}^2 \to \mathbf{R}^4$ so that the composition

$$T(\mathbf{x}) = T_2(T_1(\mathbf{x})) = (T_2 \circ T_1)(\mathbf{x})$$

is one-to-one, OR explain why this is impossible. If you provide an example, your answer should include an explanation of why this linear transformation has the desired properties.

(6) Find an example of
- a linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^4$, **and**
- linearly **dependent** vectors $\mathbf{u}$ and $\mathbf{v}$
- such that $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly **independent**,

OR explain why this is impossible. If you provide an example, your answer should include an explanation of why this linear transformation and these vectors have the desired properties.

(7) Let A and B be the following equivalent 4×5 matrices.

$$A = \begin{bmatrix} 1 & 0 & 3 & 1 & 1 \\ 2 & 0 & 3 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for $\mathrm{col}(A)$ that includes $\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$.

(8) Let A be a 3×3 matrix. We perform the following row operations and get to the matrix B below. (A_1 and A_2 denote the intermediate matrices between the row operations.)

$$A \overset{R_1 \leftrightarrow R_3}{\rightsquigarrow} A_1 \overset{R_2 = R_2 + 3R_1}{\rightsquigarrow} A_2 \overset{R_3 = R_3 - R_1}{\rightsquigarrow} B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute $\det(A)$.

(9) [10pts] Determine the eigenvalues and dimensions of the eigenspaces of

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

(10) [10pts] Let A be the 2×2 matrix

$$A = \begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix}.$$

The characteristic polynomial of A is $(4 - \lambda)(4 + \lambda)$. Compute an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

(11) Let A be an invertible 3×3 matrix and let B be a 3×4 matrix with nullity 2. What are the possible values for the nullity of AB?

(12) [10pts] Let A be a 2×2 matrix with 0 and 1 as eigenvalues and let $B = A^2 + A$. Determine the nullity of B. (*Hint:* $B = (A + I)A = A(A + I)$.)