MATH 208 C — FINAL EXAM — Autumn 2022

NAME: Solutions

To make it possible for Gradescope to recognize you, please write your name:

(1) **clearly** in **CAPITAL LETTERS**,
(2) **exactly on the line above**,
(3) use the **name you are registered under** for this class.

(1) Please put away all phones and earphones in your bag.
(2) There are 6 problems.
(3) Show all of your work and justify your answers.
(4) Write clearly.

(1) Let $S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : x_1 - 2x_2 + 3x_3 - 4x_4 = 0 \right\}$.

(a) Is S a subspace? Give reasons.

YES

Ans 1: S is the nullspace of $\begin{bmatrix} 1 & -2 & 3 & -4 \end{bmatrix}$ and all nullspaces are subspaces.

Ans 2: ① $(0,0,0,0) \in S$ since $0 - 2\cdot 0 + 3 \cdot 0 - 4 \cdot 0 = 0$

② If $x, y \in S$ then $(x_1+y_1) - 2(x_2+y_2) + 3(x_3+y_3) - 4(x_4+y_4)$
$= (x_1 - 2x_2 + 3x_3 - 4x_4) + (y_1 - 2y_2 + 3y_3 - 4y_4) = 0 + 0 = 0$

③ If $x \in S$ and $c \in \mathbb{R}$ then $cx \in S$ since
$cx_1 - 2cx_2 + 3cx_3 - 4cx_4 = c(x_1 - 2x_2 + 3x_3 - 4x_4) = 0$

(b) Find a basis for S.

$$S = \left\{ \begin{pmatrix} 2u - 3v + 4w \\ u \\ v \\ w \end{pmatrix} : u, v, w \in \mathbb{R} \right\}$$

$$= \text{Span} \left\{ \underbrace{\underset{u_1}{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{u_2}{\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{u_3}{\begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix}}}_{\text{basis of } S} \right\}$$

(c) Express the equation $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$ as $\det(M) = 0$ for a matrix M.

$$\det \begin{bmatrix} x_1 & 2 & -3 & 4 \\ x_2 & 1 & 0 & 0 \\ x_3 & 0 & 1 & 0 \\ x_4 & 0 & 0 & 1 \end{bmatrix} = 0$$

(d) Explain your logic in (c).

$S = \text{Span}\{u_1, u_2, u_3\}$. $\{u_1, u_2, u_3\}$ is a basis of S.

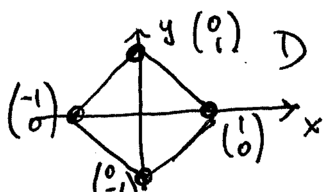
If $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ is any ~~other~~ point on S

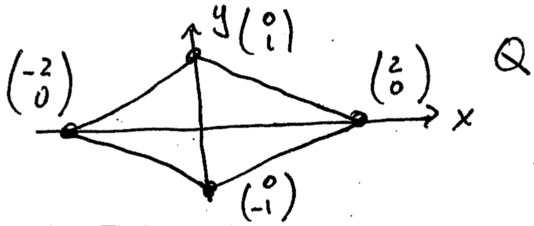
then $\{x, u_1, u_2, u_3\}$ are LD

$\therefore$ $\det \underbrace{\begin{bmatrix} x & u_1 & u_2 & u_3 \end{bmatrix}}_{M} = 0$ this is a linear condition on pts x in S

But S is a single 3d plane in $\mathbb{R}^4$ and so has one linear condition defining it. So it must be that $\det(M) = 0$ is a multiple of $x_1 - 2x_2 + 3x_4 - 4x_4 = 0$

(2) Let D be the diamond in $\mathbb{R}^2$ with corners $(-1,0),(0,1),(1,0),(0,-1)$ and let Q be the quadrilateral with corners $(-2,0),(0,1),(2,0),(0,-1)$. Draw D and Q to help answer the following.





(a) Compute the linear transformation T that takes D to Q. Write it fully with domain, codomain and map.

$$T: \mathbb{R}^2 \to \mathbb{R}^2$$
$$\begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}2\\0\end{pmatrix}$$
$$\begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}0\\1\end{pmatrix}$$

$$\therefore \quad T: \mathbb{R}^2 \to \mathbb{R}^2$$
$$\begin{pmatrix}x\\y\end{pmatrix} \mapsto \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

(b) Is T invertible? If yes, find its inverse (written fully). If not, say why not.

YES $\quad \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix} = \begin{bmatrix}1/2 & 0\\0 & 1\end{bmatrix}$

inverse exists

$$\therefore \quad T^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$$
$$\begin{pmatrix}x\\y\end{pmatrix} \mapsto \begin{bmatrix}1/2 & 0\\0 & 1\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

(3) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that projects onto the (x_1, x_2)-plane.

(a) Write down the matrix of T, i.e., A such that $T(x) = Ax$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Calculate all eigenvalues and eigenspaces of A. Explain clearly and box your answers. (No long computations needed. Think about what projection does.)

T sends any v on the (x_1, x_2)-plane to itself.

- So (x_1, x_2)-plane is in the eigenspace of eigenvalue 1. Basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \Rightarrow \lambda = 1$ with multiplicity 2

- T sends the x_3 axis to 0 so x_3-axis is in the eigenspace of eigenvalue 0. Basis $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \Rightarrow \lambda = 0$ w/ multiplicity 1

Since these spaces account for 3 eigenvalues we are done.

$$E_1 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$E_0 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(c) Diagonalize A as PDP^{-1} if possible. Identify P and then leave P^{-1} symbolic.

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P^{-1}}$$

(4) Let A be the following matrix.

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 1 \\ 0 & 1 & 5 & -6 & 2 \\ 1 & 3 & 4 & -3 & 3 \end{bmatrix}$$

(a) If $T(x) = Ax$, then what is the domain and codomain of T?

$$T: \underbrace{\mathbb{R}^5}_{\text{domain}} \longrightarrow \underbrace{\mathbb{R}^3}_{\text{codomain}}.$$

(b) Is T onto? If yes, say why. If not, find a vector b that is not in the range of T.

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 1 \\ 0 & 1 & 5 & -6 & 2 \\ 1 & 3 & 4 & -3 & 3 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_3 \leftarrow} \begin{bmatrix} 1 & 2 & -1 & 3 & 1 \\ 0 & 1 & 5 & -6 & 2 \\ 0 & 1 & 5 & -6 & 2 \end{bmatrix}$$

$$\xrightarrow[R_3 - R_2]{R_3 \leftarrow} \begin{bmatrix} 1 & 2 & -1 & 3 & 1 \\ 0 & 1 & 5 & -6 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B \text{ echelon form of } A$$

T not onto since B has a row of 0's.

Suppose $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3$. Doing the same row operations on b,

$$b \mapsto \begin{pmatrix} b_1 \\ b_2 \\ b_3 - b_1 \end{pmatrix} \longrightarrow \begin{pmatrix} b_1 \\ b_2 \\ b_3 - b_1 - b_2 \end{pmatrix}$$

$$b \notin \text{range}(T) \iff b_3 - b_1 - b_2 \neq 0$$

eg $b = \begin{pmatrix} +1 \\ +1 \\ +1 \end{pmatrix}$

(c) Compute the range of T. Say what you did.

Range (T) = colsp(A)

B has pivots in cols ① & ②

∴ Cols ① & ② of A are a basis for colsp(A)

$$\text{Colsp}(A) = \text{Span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\1\\3\end{pmatrix}\right\}$$

(d) Is T one to one? Explain.

No

$\text{Nullsp}(A) = \ker(T) = \{x : Ax = 0\} = \{x : Bx = 0\}$

Solving $Bx=0$ gives 3 free variables s, t, u

$$\text{Nullsp}(A) = \left\{\begin{pmatrix} -2(-5s+6t-2u) + s - 3t - 0 \\ -5s+6t-2u \\ s \\ t \\ u \end{pmatrix} : s, t, u \in \mathbb{R}\right\}$$

which is a 3d plane in $\mathbb{R}^5$

Since $\text{Nullsp}(A) \neq \{0\}$

T not 1-1.

(5) The following is the diagonalization of a 3×3 matrix A. Use it to answer the following questions, with reasons.

$$A = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & -1/2 \\ 0 & 0 & -1/2 \end{bmatrix}}_{P^{-1}}$$

(a) Is A invertible?

No $\quad$ 0 is an eigenvalue of A $\Rightarrow \dim E_0 = 1$

$\therefore$ Nullsp$(A) \neq \{0\}$, $\det(A) = 0$

(b) What is the rank of A?

$\text{rank}(A) = 2$ since $\underset{\dim E_0}{\underset{\parallel}{\text{nullity}}}(A) = 1$

(c) What are the coordinates of $(1, 1, 2)$ in the basis of eigenvectors associated to this diagonalization?

$$P^{-1}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & -1/2 \\ 0 & 0 & -1/2 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix}$$

(d) What is the characteristic polynomial of A^2 (up to sign)?

eigenvals of A: $0, -1, -1$

eigenvals of A^2: $0, 1, 1$

$\therefore$ Characteristic poly of A^2 is

$$\lambda(\lambda - 1)^2$$

(6) Let $A = \begin{bmatrix} -1 & 1 \\ 3 & 9 \end{bmatrix}$.

(a) Find A^{-1}.

$$A^{-1} = \frac{1}{-9-3}\begin{bmatrix} 9 & -1 \\ -3 & -1 \end{bmatrix} = \frac{1}{-12}\begin{bmatrix} 9 & -1 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} -3/4 & 1/12 \\ 1/4 & 1/12 \end{bmatrix}$$

(b) If $Ax = b$ can you write a formula for x in terms of some, or all, of A, A^{-1}, b?

$$x = A^{-1}b$$

(c) Find a quadratic of the form $y = 1 + a_1x + a_2x^2$ that passes through $(-1, 2)$ and $(3, -3)$.

$$\begin{aligned} 2 &= 1 - a_1 + a_2 \\ -3 &= 1 + 3a_1 + 9a_2 \end{aligned} \Rightarrow \begin{matrix} \cancel{2}\,1 = \\ -4 = \end{matrix} \begin{bmatrix} -1 & 1 \\ 3 & 9 \end{bmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & 9 \end{bmatrix}^{-1}\begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{bmatrix} -3/4 & 1/12 \\ 1/4 & 1/12 \end{bmatrix}\begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} \cancel{4}\, -3/4 - 1/3 \\ 1/4 - 1/3 \end{pmatrix} = \begin{pmatrix} \frac{-13}{12} \\ \frac{-1}{12} \end{pmatrix}$$

$$\boxed{y = 1 - \frac{13}{12}x - \frac{1}{12}x^2}$$

(d) Do you expect more than one such quadratic? Why?

No $Ax = b$ has a unique sol^{n} in this case

$$x = A^{-1}b$$

SCRATCH PAPER

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Points

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(b) Find a basis for S.

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(c) Express the equation $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$ as $\det(M) = 0$ for a matrix M.

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(d) Explain your logic in (c).

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$1+3+1$

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4 → eigenvalues of A^2
→ char poly of A^2

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