Your Name

Your Signature

Student ID #

**Honor Statement**

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

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- Turn off all cell phones, pagers, radios, mp3 players, and other similar devices.
- This exam is closed book. You may use one $8.5'' \times 11''$ sheet of handwritten notes (both sides OK).
  Do not share notes. No photocopied materials are allowed.
- Only the TI 30X IIS calculators is allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 9 pages, plus this cover sheet. Please make sure that your exam is complete.

| Question | Points | Score |
|---|---|---|
| 1 | 24 | |
| 2 | 14 | |
| 3 | 12 | |
| 4 | 10 | |
| 5 | 12 | |
| 6 | 18 | |
| Total | 90 | |

1. (24 points) Indicate whether the given statement is true or false (2 pts) and give justification as to why it is true or false(2 pts).

a) [4 pt] If $A$ is invertible and $A^2 - A$ is not invertible, then 1 is an eigenvalue of $A$.

TRUE. If $A^2 - A$ is not invertible, then $\det(A^2 - A) = 0$. Furthermore, since $A^2 - A = A(A - I_n)$ applying the determinant gives us the following equation

$$\det(A(A - I_n)) = \det(A)\det(A - I_n) = 0$$

Since $\det(A) \neq 0$, we must have that $\det(A - I_n) = 0$, which means exactly that 1 is an eigenvalue of $A$.

b) [4 pts] If $A$ is an $n \times n$ matrix, and $r \in \mathbb{R}$ is a non-zero scalar, then $\det(rA) = r\det(A)$. (Hint: Think about what multiplying **one** row of a matrix by a scalar does to the determinant.)

FALSE. Recall that for a matrix $A$, if we apply the row operation $rR_i \to R_i$ (multiplying a row by a constant $r$), the determinant changes via $det(A) \to rdet(A)$. Now, the matrix $rA$ is $\begin{bmatrix} ra_{11} & ra_{12} & \cdots & ra_{1n} \\ ra_{21} & ra_{22} & \cdots & ra_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ra_{n1} & \cdots & \cdots & ra_{nn} \end{bmatrix}$ so this is just multiplying each row by $r$ and since there are $n$ rows, the determinant will be scaled by $r \cdot r \cdots r$, n times. This means that $\det(rA)$=$r^n\det(A)$.

c) [4 pts] If $A$ is an $n \times n$ matrix and $A$ is diagonalizable, then $A^k$ is diagonalizable for $k = 1, 2, \ldots$

TRUE. If $A$ is diagonalizable, then there exists and invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$. In computing $A^k$ we see that

$$A^k = PDP^{-1}PDP^{-1} \cdots PDP^{-1} = PD^kP^{-1}$$

Since $D^k$ is a diagonal matrix, we've just written $A^k$ as $PD'P^{-1}$ where $D'$ is diagonal, hence $A^j$ is diagonalizable.

Give an example of each of the following. If it is not possible write "NOT POSSIBLE", and **give justification as to why**.

d) [2 pts] A $2 \times 2$ matrix $A$ with eigenvalues and eigenvectors $\lambda_1 = 1, \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\lambda_2 = 3$, $\vec{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Recall from the process of diagonalizing a matrix $A$, we choose $P$ to have columns as the eigenvalues of $A$, and $D$ to have eigenvalues on the diagonal. Since these are all given to us, we let $P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. Computing $P^{-1}$ we see that $P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Now for $A = PDP^{-1}$, we know $A$ has the desired eigenvalues and eigenvectors, hence

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$$

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e) [2 pts] A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T$ takes $S$ to a rectangle of area 7, where $S$ is taken to be the unit square lying in the first quadrant. Giving the matrix of the transformation (if it exists) is sufficient.

There are many possibilities for this but the two easiest ones are

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix}$$

Recall that the first column is always where $\vec{e}_1$ goes and the second column is always where $\vec{e}_2$ goes. What these matrices to do the unit square should be clear from this argument.

f) [2 pts] A 2-dimensional subspace of $\mathbb{R}^3$ that includes the vector $(-1, 3, -2)$. (Hint: What does any 2-dimensional subspace of $\mathbb{R}^3$ look like and can you express it with one equation?)

Any two-dimensional subspace of $\mathbb{R}^2$ is plane and planes are given by a single equation in $x, y$, and $z$. This reduces this question to finding any equation in $x, y$, and $z$ containing the given point. The easiest one is $x + y + z = 0$.

The following are short answer questions. Please state your answer and give justification as to how you arrived at your solution.

g) [3 pts] Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be an **onto** linear transformation, with $m > n$. What is $\dim(\ker(T))$?

> Since $T$ is onto, dim(range($T$))$= n$. By rank-nullity, dim(range($T$))+dim(ker($T$))$= m$, hence n+dim(ker($T$))$= m$ implies dim(ker($T$))$= m - n$.

h) [3 pts] Let $A$ be an $n \times n$ matrix. If $\lambda$ is an eigenvalue of $A$, with eigenvector $\vec{v}$, what is an eigenvalue of $A^2$?

> If $\lambda$ is an eigenvalue of $A$, with eigenvector $\vec{v}$, then $A\vec{v} = \lambda\vec{v}$. Considering $A^2\vec{v}$ we have that
> $$A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda(A\vec{v}) = \lambda(\lambda\vec{v}) = \lambda^2\vec{v}$$
> This means that if $\lambda$ is an eigenvalue of $A$ then $\lambda^2$ is en eigenvalue of $A^2$.

2. (14 points) Let $A = \begin{bmatrix} -1 & -3 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. The characteristic polynomial of $A$ is $-\lambda^3 + 4\lambda^2 - 4\lambda$.

a) [10 pts] Find the eigenvalues of $A$ and a basis for each eigenspace. (Hint: You may find it easiest to factor a $-\lambda$ out of the characteristic polynomial).

The characteristic polynomial factors as $-\lambda^3 + 4\lambda^2 - 4\lambda = -\lambda(\lambda^2 - 4\lambda + 4) = -\lambda(\lambda - 2)^2$ so our eigenvalues are 0 and 2 with multiplicities 1 and 2 respectively. Computing eigenspaces we look at the usual null spaces:

$$A - 0I_3 = \begin{bmatrix} -1 & -3 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} -1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

so we can see that $x_3 = 0, x_2 = s$ and $x_1 = -3s$ hence we have

$$E_0 = span\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\}$$

. For $\lambda = 2$ we have

$$A - 2I_3 = \begin{bmatrix} -3 & -3 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -3 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so $x_3 = 0, x_2 = s$ and $x_1 = -s$ hence

$$E_0 = span\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

b) [4 pts] Is $A$ diagonalizable? If so, give an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$. If not, explain why $A$ cannot be diagonalizable.

$A$ is not diagonalizable because we don't have a basis of eigenvectors.

3. (12 points) Consider the following matrix and its reduced echelon form below

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 1 \\ 1 & -1 & -2 & -1 & 1 & 2 \\ 2 & -2 & -4 & -2 & 1 & 2 \\ -1 & 1 & 2 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & 4 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) [3 pts] Find a basis for col($A$)

Applying recipe 2 we get that

$$\mathcal{B}_{col(A)} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

b) [3 pts] Find a basis for row($A$)

Applying recipe 1 we get that

$$\mathcal{B}_{row(A)} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

c) [4pts] Find a basis for Null($A$) and determine rank($A$) and nullity($A$).

Translating the above matrix into a linear system we see that $x_3 = s_1, x_4 = s_2$, and $x_6 = s_3$. Moreover, $x_5 = -2s_3$, $x_2 + 4s_1 + 2s_2 - s_3 = 0 \implies x_2 = -4s_1 - 2s_2 + s_3$. Finally, $x_1 = -2s_1 - s_2 - s_3$ This means our basis is

$$\mathcal{B}_{Null(A)} = \left\{ \begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$Rank(A) = Nullity(A) = 3.$

d) [2 pts] Let $T_A : \mathbb{R}^6 \to \mathbb{R}^4$ be given by $T_A(\vec{x}) = A\vec{x}$. Determine if $T_A$ is one-to-one, onto, and/or invertible. Justify why $T_A$ has or doesn't have these properties.

$T_A$ is clearly not invertible because its not square. $T_A$ is onto $\Leftrightarrow$ rank($A$) $= 4$, which we can see if false. Moreover, since $6 < 4$,$T_A$ is not one-to-one.

4. (10 points) Let $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and consider the basis of $\mathbb{R}^2$ given by $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

a) [4 pts] Find the coordinate vector $[\vec{x}]_{\mathcal{B}}$ of $\vec{x}$ with respect to the basis $\mathcal{B}$. (Note that $\vec{x}$ is given in the standard basis.)

The change of basis matrix is given by $U = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$. $U^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ and since $U^{-1}$ takes vectors in the standard basis and expresses them in the basis $\mathcal{B}$, we obtain $[\vec{x}]_{\mathcal{B}}$ by computing

$$[\vec{x}]_{\mathcal{B}} = U^{-1}\vec{x} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

b) [6pts] Let $\mathcal{C}$ be another basis for $\mathbb{R}^2$ and let the change of basis matrix **from** $\mathcal{C}$ **to** $\mathcal{B}$ be given by $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$. Find the coordinate vector $[\vec{x}]_{\mathcal{C}}$ of $\vec{x}$ with respect to $\mathcal{C}$. (CAUTION: The change of basis matrix goes from a non-standard basis to another non-standard basis, so the matrix needed to compute $[\vec{x}]_{\mathcal{C}}$ needs to be found. You may find it helpful to draw the usual picture for changing between non-standard bases, and see where the matrix you need fits into the picture.)

If we let $V$ be the change of basis matrix taking elements in the basis $\mathcal{C}$ to elements in the standard basis, then with $U$ as the change of basis matrix from above, we have that the change of basis matrix from $\mathcal{C}$ to $\mathcal{B}$ first takes $\mathcal{C}$ to the standard basis, then takes the standard basis to $\mathcal{B}$. In terms of matrices, this cooresponds to applying $V$ first, then $U^{-1}$ hence this matrix is given by $U^{-1}V$.

Since this matrix is given to us, we know that $U^{-1}V = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$. Our aim is to express $\vec{x}$ in the basis $\mathcal{C}$. This is just given by $V^{-1}\vec{x}$ so we must find $V^{-1}$. Using the above matrix equation we can see that multiplying on the left by $U$ gives $V = U \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and its clear that $V = V^{-1}$ hence $[\vec{x}]_{\mathcal{C}} = V^{-1}\vec{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

5. (12 points) Determine if the following are subspaces. If they are, show why. If they're not, give justification as to what subspace condition breaks.

   a) [6 pts] $S = \left\{ \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n : v_1 + v_2 + \cdots + v_n = 0 \right\}$. That is, $S$ is the set of all vectors in $\mathbb{R}^n$ whose components sum to 0.

   > $S$ is a subspace because $S$ is the null space of the $1 \times n$ matrix $\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$. That is, $S = Null(\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix})$.

   b) [6pts] $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x = y = 0 \quad \text{or} \quad x \neq y \right\}$. That is, $S$ consists of all vectors in $\mathbb{R}^2$ where both coordinates are never equal, unless they're both zero. If you like to think geometrically, this is all of $\mathbb{R}^2$ with the line $y = x$ removed, except the origin $(0, 0)$.

   > $S$ is not a subspace because it is not closed under vector addition. Take $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. It's easy to see that $\vec{u}, \vec{v} \in S$ but their sum $\vec{u} + \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in $S$ since both coordinates are equal.

6. (18 points) Let $A$ be a $5 \times 5$ matrix and assume it has the following eigenspaces (where the vectors written below form a basis for each eigenspace, these vectors are NOT the columns of $A$).

$$E_1 = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\},\ E_2 = span\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\},\ E_0 = span\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\},\ E_{-1} = span\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

a) [3 pts] Give a basis for Null($A$) (Hint: Find the relationship between the null space and one of the eigenspaces)

Since $E_0 = Null(A - 0I_5) = Null(A)$, $\mathcal{B}_{Null(A)} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

b) [3 pts] Let $T_A(\vec{x}) = A\vec{x}$. Is $T_A$ one-to one, onto, and/or invertible? Give justification for each property.

Since Nullity($A$) $= 1 \neq 0$, by the big theorem, $T_A$ is not one-to-one. This also implies it is not onto since it must be neither or both. Again, by the big theorem, since $\ker(T_A) \neq \{0\}$, $T_A$ is not invertible, hence $A$ is not invertible.

c) [3 pts] What is rank($A - 2I_5$) and why?

Since dim($E_2$) =dim(Null(($A - 2I_5$)))= 2, rank-nullity implies that rank($A - 2I_5$) = 3.

d) [3 pts] Is $A$ diagonalizable? Explain why it is or isn't.

$A$ is diagonalizable. Since $A$ is a $5 \times 5$ matrix and has 5 eigenvectors, together they form a basis of eigenvectors, which is equivalent to diagonalizability.

e) [6 pts] Find all vectors $\vec{x} \in \mathbb{R}^5$ such that $T_A(\vec{x}) = -\vec{x}$. (Hint: Try to rewrite this set of vectors as the null space of a certain matrix and relate it to eigenspaces)

Any vector satisfying $T_A(\vec{x}) = A\vec{x} = -\vec{x}$ satisfies

$$A\vec{x} + I_5\vec{x} = (A + I_5)\vec{x} = \vec{0}$$

hence any vector like this lives in Null($A + I_5$) $= E_{-1}$. Since we are given a basis for $E_{-1}$, it follows that any vector like the one above must be a multiple of $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.