Your Name

Your Signature

Student ID #

Honor Statement

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________

- Turn off all cell phones, pagers, radios, mp3 players, and other similar devices.
- This exam is closed book. You may use one $8.5'' \times 11''$ sheet of handwritten notes (both sides OK).
  Do not share notes. No photocopied materials are allowed.
- Only the TI 30X IIS calculators is allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 9 pages, plus this cover sheet. Please make sure that your exam is complete.

Question	Points	Score
1	24	
2	14	
3	12	
4	10	
5	12	
6	18	
Total	90	

1. (24 points) Indicate whether the given statement is true or false (2 pts) and give justification as to why it is true or false(2 pts).

   a) [4 pt] If A is an invertible $n \times n$ matrix and $A^2 - A$ is not invertible, then 1 is an eigenvalue of A. (Hint: Try factoring out an A from the matrix equation)

   b) [4 pts] If A is an $n \times n$ matrix, and $r \in \mathbb{R}$ is a non-zero scalar, then $\det(rA) = r\det(A)$. (Hint: Think about what multiplying **one** row of a matrix by a scalar does to the determinant.)

   c) [4 pts] If A is an $n \times n$ matrix and A is diagonalizable, then A^k is diagonalizable for $k = 1, 2, \ldots$

Give an example of each of the following. If it is not possible write "NOT POSSIBLE", and **give justification as to why**.

d) [2 pts] A 2×2 matrix A with eigenvalues and eigenvectors $\lambda_1 = 1, \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\lambda_2 = 3$, $\vec{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

e) [2 pts] A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that T takes S to a rectangle of area 7, where S is taken to be the unit square lying in the first quadrant. Giving the matrix of the transformation (if it exists) is sufficient.

f) [2 pts] A 2-dimensional subspace of $\mathbb{R}^3$ that includes the vector $(-1, 3, -2)$. (Hint: What does any 2-dimensional subspace of $\mathbb{R}^3$ look like and can you express it with one equation?)

The following are short answer questions. Please state your answer and give justification as to how you arrived at your solution.

g) [3 pts] Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be an **onto** linear transformation, with $m > n$. What is $\dim(\ker(T))$?

h) [3 pts] Let A be an $n \times n$ matrix. If λ is an eigenvalue of A, with eigenvector $\vec{v}$, what is an eigenvalue of A^2?

2. (14 points) Let $A = \begin{bmatrix} -1 & -3 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. The characteristic polynomial of A is $-\lambda^3 + 4\lambda^2 - 4\lambda$.

   a) [10 pts] Find the eigenvalues of A and a basis for each eigenspace. (Hint: You may find it easiest to factor a $-\lambda$ out of the characteristic polynomial).

   b) [4 pts] Is A diagonalizable? If so, give an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If not, explain why A cannot be diagonalizable.

3. (12 points) Consider the following matrix and its reduced echelon form below

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 1 \\ 1 & -1 & -2 & -1 & 1 & 2 \\ 2 & -2 & -4 & -2 & 1 & 2 \\ -1 & 1 & 2 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & 4 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) [3 pts] Find a basis for col(A)

b) [3 pts] Find a basis for row(A)

c) [4pts] Find a basis for Null(A) and determine rank(A) and nullity(A).

d) [2 pts] Let $T_A : \mathbb{R}^6 \to \mathbb{R}^4$ be given by $T_A(\vec{x}) = A\vec{x}$. Determine if T_A is one-to-one, onto, and/or invertible. Justify why T_A has or doesn't have these properties.

4. (10 points) Let $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and consider the basis of $\mathbb{R}^2$ given by $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

   a) [4 pts] Find the coordinate vector $[\vec{x}]_{\mathcal{B}}$ of $\vec{x}$ with respect to the basis $\mathcal{B}$. (Note that $\vec{x}$ is given in the standard basis.)

   b) [6pts] Let $\mathcal{C}$ be another basis for $\mathbb{R}^2$ and let the change of basis matrix **from** $\mathcal{C}$ **to** $\mathcal{B}$ be given by $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$. Find the coordinate vector $[\vec{x}]_{\mathcal{C}}$ of $\vec{x}$ with respect to $\mathcal{C}$. (CAUTION: The change of basis matrix goes from a non-standard basis to another non-standard basis, so the matrix needed to compute $[\vec{x}]_{\mathcal{C}}$ needs to be found. You may find it helpful to draw the usual picture for changing between non-standard bases, and see where the matrix you need fits into the picture.)

5. (12 points) Determine if the following are subspaces. If they are, show why. If they're not, give justification as to what subspace condition breaks.

a) [6 pts] $S = \left\{ \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n : v_1 + v_2 + \cdots + v_n = 0 \right\}$. That is, S is the set of all vectors in $\mathbb{R}^n$ whose components sum to 0. (Hint:For any matrix A, Null(A) is a subspace).

b) [6pts] $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x = y = 0 \quad \text{or} \quad x \neq y \right\}$. That is, S consists of all vectors in $\mathbb{R}^2$ where both coordinates are never equal, unless they're both zero. If you like to think geometrically, this is all of $\mathbb{R}^2$ with the line $y = x$ removed, and the origin $(0, 0)$, put back in.

6. (18 points) Let A be a 5×5 matrix and assume it has the following eigenspaces (where the vectors written below form a basis for each eigenspace, these vectors are NOT the columns of A).

$$E_1 = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}, \quad E_2 = span\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad E_0 = span\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad E_{-1} = span\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

a) [3 pts] Give a basis for Null(A) (Hint: Find the relationship between the null space and one of the eigenspaces)

b) [3 pts] Let $T_A(\vec{x}) = A\vec{x}$. Is T_A one-to one, onto, and/or invertible? Give justification for each property.

c) [3 pts] What is $\text{rank}(A - 2I_5)$ and why?

d) [3 pts] Is A diagonalizable? Explain why it is or isn't.

e) [6 pts] Find all vectors $\vec{x} \in \mathbb{R}^5$ such that $T_A(\vec{x}) = \vec{x}$. (Hint: Try to rewrite this set of vectors as the null space of a certain matrix and relate it to eigenspaces)