Linear Programming

Rekha R. Thomas

Linear programming is the problem of optimizing a linear function over the solutions to a finite set of linear inequalities. It has a wide array of applications in real life and extremely sophisticated software packages for its solution that routinely solve large problems with hundreds of thousands, even a million variables. Applications of linear programming come from all walks of life including the food industry, transportation (bus, airline, train, ship scheduling), finance, natural sciences, social sciences, classroom scheduling, manufacturing, agriculture etc. Linear programming is built on linear algebra and is one of the most showy and useful applications of it.

Below we examine one of the oldest problems in linear programming, the *Diet Problem* which was made famous during World War II. For background on this problem and an example, see `https://neos-guide.org/content/diet-problem`. This site has a fun solver you can link to, to solve your own diet problem.

Problem 1. Some athletes are very concerned with maintaining or increasing their weight in order to be ready for the next season. In the summer, YZ decides to bulk up on whole milk chocolate milk and power bars. Both food items can be sickening if consumed in very large quantities, so YZ would like to limit his intake to at most 3 cups of milk and 3 bars per day. The nutrition facts on his chosen foods are as follows.

Nutrition Facts
Serving Size: 1 cup

Amount Per Serving	
Calories 208	Calories from Fat 76
	% Daily Values*
Total Fat 8.48g	**13%**
Saturated Fat 5.26g	**26%**
Polyunsaturated Fat 0.31g	
Monounsaturated Fat 2.475g	
Cholesterol 30mg	**10%**
Sodium 150mg	**6%**
Potassium 418mg	
Total Carbohydrate 25.85g	**9%**
Dietary Fiber 2g	**8%**
Sugars 23.85g	
Protein 7.92g	
Vitamin A 5%	Vitamin C 4%
Calcium 28%	Iron 3%

* Percent Daily Values are based on a 2000 calorie diet. Your daily values may be higher or lower depending on your calorie needs.

Nutrition Facts
Serving Size: 1 bar (93g)

Amount Per Serving	
Calories 330	Calories from Fat 81
	% Daily Values*
Total Fat 9g	**14%**
Saturated Fat 4.5g	**22%**
Trans Fat 0g	
Cholesterol 4mg	**1%**
Sodium 180mg	**8%**
Potassium 300mg	
Total Carbohydrate 40g	**13%**
Dietary Fiber 5g	**20%**
Sugars 27g	
Protein 30g	
Vitamin A 0%	Vitamin C 0%
Calcium 8%	Iron 8%

* Percent Daily Values are based on a 2000 calorie diet. Your daily values may be higher or lower depending on your calorie needs.

YZ does not want to take in more than 24% of the recommended daily dose of iron and would like to get at least 40% of his daily fiber needs from these food items.

1. What is the maximum number of calories that YZ can get in a day?
2. What combination of foods gets him that maximum number of calories?
3. Is it possible for YZ to have a different combination of foods and still get the same number of calories?

Solution. 1. The first step is to model the problem which involves choosing the variables and writing down the constraints and the objective function.

Let m be the number of cups of milk YZ drinks and b be the number of bars that YZ eats in a day. Then the objective is to maximize $208m + 330b$.

YZ wants to limit both servings to at most three. So, $0 \le m \le 3$ and $0 \le b \le 3$. The iron constraint says that $3m + 8b \le 24$ and the fiber constraint says that $8m + 20b \ge 40$. So altogether, the problem YZ needs to solve is

$$\begin{array}{ll} \text{maximize} & 208m + 330b \\ \text{subject to} & 0 \le b \le 3 \\ & 0 \le m \le 3 \\ & 3m + 8b \le 24 \\ & 8m + 20b \ge 40 \end{array}$$

First of, this is an optimization problem since we are optimizing (maximizing in this case) a function over all solutions to a set of constraints. In particular, this is a linear program since all functions involved are linear functions of the two variables m and b.

2. The next step is to draw the set of all solutions. For this we plot the regions specified by the constraints. Each of these regions is one side of a line in $\mathbb{R}^2$. Such a region is called a *halfspace* in $\mathbb{R}^2$ since it cuts $\mathbb{R}^2$ into two halves. The set of solutions is the common intersection of the six halfspaces in the problem.

Note that the *feasible region*, namely, the set of all solutions, is a "box" with flat sides. Such a shape is called a *polyhedron*. Note that each point (x, y) in the feasible region is one possible choice of milk servings and bar servings that meet all the constraints. For example, the solution $(1.5, 1.5)$ is in the feasible region which says that YZ could drink 1.5 cups of milk and 1.5 bars a day to satisfy all constraints. The amount of calories this would give him is $1.5 \times 208 + 1.5 \times 330 = 807$. The iron intake would be 16.5% of daily needs and the fiber intake would be 42% which meets all rules.

3. Can you find a plan better than $(1.5, 1.5)$? For instance, check that $(0, 3)$ is a much better plan in terms of increasing calories since that would give him 990 calories and 60% of his required daily fiber. This plan is at the brink of his iron rule since it would give him exactly 24% of daily iron. Is there an even better solution?

4. To answer the last question, we calculate the four corners of the feasible region. This is where linear algebra comes in. Each corner is the unique solution to two of the six linear equations that specify the boundary of the feasible region. In our case the corners are $(0, 3), (0, 2), (3, \frac{15}{8}), (3, \frac{4}{5})$. All are feasible solutions, so satisfy all constraints. Among these, the most calories come from $(3, \frac{15}{8})$, namely 1242.75 per day. This plan has YZ consuming 3 cups of milk and almost 2 bars each day.

5. Is there an even better solution than the corner $(3, \frac{15}{8})$? In fact, no since the linear function $208m + 330b$ will always maximize (or minimize) at a corner of the box like feasible region. Why is this? Plot the line $208m + 330b = 0$. This is a line in $\mathbb{R}^2$ passing through the origin. If you translate this line parallel to itself to the ++ quadrant, the right hand side value increases. For example, if you translate the line to pass through the corner $(0, 2)$, then the value of the right hand side is $208 \times 0 + 330 \times 2 = 660$ which is larger than 0. Solving the linear program involves pushing the line in the direction in which the right hand side increases until you hit the last possible point in the feasible region. This point is precisely $(3, \frac{15}{8})$ and this is thus the optimal plan. There are no other optimal plans.

1. What all did linearity buy you in this problem? A feasible region with flat sides, an algorithm for solving the problem which boils down to "pushing" a line across the feasible region until you hit the last point in the region. Think through this.

2. What happens in higher dimensions? For example, suppose YZ also wanted to add cashew nuts to his menu. How does the geometry of the problem change?

3. Linear programs have famous algorithms, the first of which is the *simplex method* that moves from corner to corner of the feasible region so that the objective function is monotonically increasing (at least never decreasing) as you move along. This algorithm is fundamentally based in linear algebra and is considered to be one of the top 10 algorithms of the 20th century. If you plan to take a class in linear programming (such as Math 407), it is crucial to understand the geometry in linear algebra.