Solutions

MATH 207 O
MIDTERM 2
AUTUMN 2021

Name ______________________ Student ID # ____________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE: ______________________

1	10	
2	10	
3	10	
4	8	
5	12	
Total	50	

- Pace yourself. You have 50 minutes to complete the exam.
- Do not write within 1 cm of the edges of each page as these exams will be scanned.
- If you need more space, you may use the back of the page but be sure to clearly indicate that you have done so.
- Unless otherwise indicated, show all your work and justify your answers.
- You may use the Texas Instruments TI-30X IIS calculator and a 8.5×11-inch double-sided sheet of handwritten notes. All other electronic devices (including graphing and programmable calculators and calculators with calculus functions) are not permitted.
- The use of headphones or earbuds during the exam is not permitted.
- You are not allowed to use your phone for any reason during this exam. Turn your phone off and put it away for the duration of the exam.
- Raise your hand if you have any questions.

GOOD LUCK!

1. (a) (6 points) Find the general solution of the differential equation:

$$y'' - 10y' + 29y = 0$$

$r^2 - 10r + 29 = 0$

$r^2 - 10r + 25 + 4 = 0$

$(r-5)^2 = -4$

$r = 5 \pm 2i$

$$\boxed{y = C_1 e^{5t}\cos(2t) + C_2 e^{5t}\sin(2t)}$$

(b) (4 points) Find the unique solution that satisfies the initial conditions:

$$y(0) = \frac{5}{2}, \qquad y'(0) = \frac{1}{2}$$

$y(0) = C_1 e^0 \cos(0) + C_2 e^0 \sin(0) = C_1 = \frac{5}{2}$

$y'(t) = 5C_1 e^{5t}\cos(2t) - 2C_1 e^{5t}\sin(2t)$
$\quad + 5C_2 e^{5t}\sin(2t) + 2C_2 e^{5t}\cos(2t)$

$y'(0) = 5C_1 e^0 \cos(0) - 2C_1 e^0 \sin(0)$
$\quad + 5C_2 e^0 \sin(0) + 2C_2 e^0 \cos(0)$
$= 5C_1 + 2C_2 = \frac{1}{2}$

$\frac{25}{2} + 2C_2 = \frac{1}{2}$

$2C_2 = -12$

$C_2 = -6$

$$\boxed{y = \frac{5}{2} e^{5t}\cos(2t) - 6e^{5t}\sin(2t)}$$

2. (a) (4 points) Find a particular solution to

$$y'' - y' - 2y = 5\sin(2t)$$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r = 2, -1$

$y_c = c_1 e^{2t} + C_2 e^{-t}$

Initial guess:

$y_p = A\cos(2t) + B\sin(2t)$

Modify? No

$y_p' = -2A\sin(2t) + 2B\cos(2t)$

$y_p'' = -4A\cos(2t) - 4B\sin(2t)$

$(-4A\cos(2t) - 4B\sin(2t)) - (-2A\sin(2t) + 2B\cos(2t))$
$- 2(A\cos(2t) + B\sin(2t)) = 5\sin(2t)$

$(-6A - 2B)\cos(2t) + (2A - 6B)\sin(2t) = 5\sin(2A)$

$-6A - 2B = 0$ $\quad$ $2A - 6B = 5$

$B = -3A$ $\quad$ $2A + 18A = 5$

$B = -\frac{3}{4}$ $\quad$ $20A = 5$ $\quad$ $A = \frac{1}{4}$

$\boxed{y_p = \frac{1}{4}\cos(2t) - \frac{3}{4}\sin(2t)}$

(b) (4 points) Find a particular solution to

$$y'' - y' - 2y = 3e^t - e^{-t}$$

Initial guess

$y_p = Ae^t + Be^{-t}$

Modify? Yes, e^{-t} is in y_c

$y_p = Ae^t + Bte^{-t}$

$y_p' = Ae^t + Be^{-t} - Bte^{-t}$

$y_p'' = Ae^t - Be^{-t} - Be^{-t} + Bte^{-t}$
$= Ae^t - 2Be^{-t} + Bte^{-t}$

$(Ae^t - 2Be^{-t} + Bte^{-t}) - (Ae^t + Be^{-t} - Bte^{-t})$
$-2(Ae^t + Bte^{-t}) = 3e^t - e^{-t}$

$-2Ae^t + (-3B)e^{-t} = 3e^t - e^{-t}$

$-2A = 3$ $\quad$ $-3B = -1$

$A = -\frac{3}{2}$ $\quad$ $B = \frac{1}{3}$

$\boxed{y_p = -\frac{3}{2}e^t + \frac{1}{3}te^{-t}}$

(c) (2 points) Give the general solution to

$$y'' - y' - 2y = 5\sin(2t) + 3e^t - e^{-t}$$

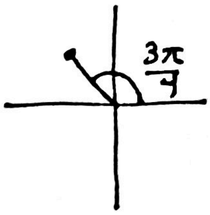
$y = y_c + y_p$

$\boxed{y = c_1 e^{2t} + c_2 e^{-t} + \frac{1}{4}\cos(2t) - \frac{3}{4}\sin(2t) - \frac{3}{2}e^t + \frac{1}{3}e^{-t}}$

3. (a) (6 points) A spring-mass system has the following equation of motion:

$$u(t) = -4\cos(5t) + 4\sin(5t), \qquad t \geq 0.$$

i. Rewrite $u(t)$ in the form $R\cos(\omega t - \varphi)$.



$R = \sqrt{(-4)^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$

$\omega = 5$

$\cos\varphi = \frac{-4}{4\sqrt{2}} = -\frac{\sqrt{2}}{2} \qquad \sin\varphi = \frac{4}{4\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\varphi = \frac{3\pi}{4}$

$\boxed{u(t) = 4\sqrt{2}\cos\left(5t - \frac{3\pi}{4}\right)}$

ii. Find the first time the mass passes through its equilibrium position.

$u(t) = 0$ Want smallest positive t so that

$\cos\left(5t - \frac{3\pi}{4}\right) = 0$

$5t - \frac{3\pi}{4} = -\frac{\pi}{2}$

$5t = \frac{\pi}{4} \qquad \boxed{t = \frac{\pi}{20}}$

(b) (4 points) A spring-mass system with external forcing function having the equation

$$3u'' + ku = 7\cos(4t)$$

experiences resonance. Find the value of k and briefly explain the main characteristics the resulting motion (max two sentences).

Resonance occurs when natural frequency equals input frequency

$\omega_0 = \sqrt{k/3} = 4$

$k/3 = 16$

$\boxed{k = 48}$

With resonance, the frequency of the system is in sync with the frequency of the forcing function. This results in energy being added to the system and consequently unbounded amplitude.

4. (8 points) Find a second order differential equation whose general solution is

$$y = \underbrace{c_1e^{3t} + c_2te^{3t}}_{y_c} + \underbrace{e^{-2t} + t^2 - 3}_{y_p}$$

Repeated real root

$r = 3$

$(r-3)^2 = 0$

$r^2 - 6r + 9 = 0$

$y'' - 6y' + 9y = 0$ ← homogeneous equation

$y'' - 6y' + 9y = F(t)$

Particular solution satisfies this equation

$y_p = e^{-2t} + t^2 - 3$

$y_p' = -2e^{-2t} + 2t$

$y_p'' = 4e^{-2t} + 2$

$F(t) = (4e^{-2t}+2) - 6(-2e^{-2t} + 2t) + 9(e^{-2t} + t^2 - 3)$

$= 25e^{-2t} + 9t^2 - 12t - 25$

$$\boxed{y'' - 6y' + 9y = 25e^{-2t} + 9t^2 - 12t - 25}$$

5. A certain spring is known to stretch 0.06 m beyond its natural length when a force of 0.54 N is applied to it. An unknown mass m is attached to this spring along with a damping device with constant $\gamma = 6$ N · sec/m. The mass is pushed up 0.04 m and released from rest. The resulting motion is observed to oscillate with decreasing amplitude and a quasi-period of $T = 4\pi/3$ sec. Use the convention that the downward direction is positive.

(a) (8 points) Find the value of m. (Hint: What is the quasi-frequency?)

Hooke's law:

$0.54 = K(0.06)$ $\gamma = 6$

$K = 9$

Quasi-frequency

$$\mu = \frac{\sqrt{4mk - \gamma^2}}{2m} \qquad \mu = \frac{2\pi}{T} = \frac{2\pi}{4\pi/3} = \frac{3}{2}$$

$$\frac{3}{2} = \frac{\sqrt{4m(9) - 6^2}}{2m}$$

$$3m = \sqrt{36m - 36}$$
$$3m = 6\sqrt{m-1}$$
$$m = 2\sqrt{m-1}$$
$$m^2 = 4m - 4$$
$$m^2 - 4m + 4 = 0$$
$$(m-2)^2 = 0$$
$$\boxed{m = 2}$$

(b) (4 points) Write down the initial value problem for the above spring-mass system (do not solve).

$$\boxed{2u'' + 6u' + 9u = 0, \quad u(0) = -0.04, \quad u'(0) = 0}$$