

Name (PRINT): _____

Student ID (PRINT): _____

Autumn 2022 – Introduction to Differential Equations Second Examination

Instructions

1. The use of all electronic devices is prohibited. Any electronic device needs to be turned off and placed in your bag. Any textbooks or notes also need to be placed in your bag.
2. Present your solutions in the space provided. Show all your work neatly and concisely. Clearly indicate your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

Scholastic dishonesty will not be tolerated and may result in terminating the midterm early. The work on this test is my own.

Signature: _____

Exercise 1. (5 points)

- Find the solution to the initial value problem

$$y'' - y = 0, \quad y(0) = 5, \quad y'(0) = -3$$

- Find the minimal value of the solution.

Key:

- The differential equation is homogeneous with constant coefficients. Let's take the characteristic equation:

$$r^2 - 1 = 0$$

2 real roots: $r = \pm 1$. The general solution is

$$y(t) = c_1 e^t + c_2 e^{-t}$$

Initial conditions: $y(0) = 5$, therefore $c_1 + c_2 = 5$.

$y'(0) = -3$, therefore $c_1 - c_2 = -3$.

By adding the 2 equations

$$2c_1 = 5 - 3 = 2 \quad \implies \quad c_1 = 1$$

Hence $c_2 = 5 - 1 = 4$.

The solution to the initial value problem is $y(t) = e^t + 4e^{-t}$

- to find the minimum, let's find where the derivative is 0. $y'(t) = e^t - 4e^{-t} = 0$,

$$e^t = 4e^{-t}$$

Divide both sides by e^{-t} ,

$$e^{2t} = 4$$

$$2t = \ln(4)$$

There is a unique value t for which the derivative is 0:

$$t = \frac{\ln(4)}{2} = \ln(2)$$

At $t = \ln(2)$, $y(\ln(2)) = e^{\ln(2)} + 4e^{-\ln(2)} = 2 + \frac{4}{2} = 4$.

Let's use the second derivative test to check whether the solution has a minimum at $t = \ln(2)$:

The differential equation is equivalent to $y'' = y$. At $t = \ln(2)$, $y''(\ln(2)) = y(\ln(2)) = 4 > 0$. By the second derivative test, the function has a minimum at $t = \ln(2)$ and the minimal value is $\boxed{4}$.

Exercise 2. (5 points) Consider the initial value problem

$$y'' + 2y' + 5y = 0, \quad y(0) = 2, \quad y'(0) = \alpha > 0$$

- Find the solution to the initial value problem.
- Find α such that $y = 0$ when $x = 1$.

Leave your answer using usual function in the form $\alpha = \dots$. No numerical approximation is required.

$\alpha = e^{-25 \frac{\text{Arctan}17}{\ln(\pi)}}$ is a valid format for your answer.

Key:

- The differential equation is homogeneous and has constant coefficients. Let's take the characteristic equation:

$$r^2 + 2r + 5 = 0$$

$$r = -1 \pm 2i$$

The general solution is

$$y(t) = c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t)$$

Initial conditions: $y(0) = 2 = c_1$.

$$y'(0) = -2 + 2c_2 = \alpha.$$

$$c_2 = \frac{\alpha+2}{2}.$$

The solution to the initial value problem is $y(t) = 2e^{-t} \cos 2t + \left(\frac{\alpha+2}{2}\right) e^{-t} \sin 2t$

- $y(1) = 0 = 2e^{-1} \cos(2) + \left(\frac{\alpha+2}{2}\right) e^{-1} \sin(2)$

$$\frac{\alpha + 2}{2} = -\frac{\cos(2)}{\sin(2)} = -\cot(2)$$

$$\alpha = -2 \cot(2) - 2$$

Exercise 3. (5 points) Given the differential equation

$$y'' + 2y' + y = xe^{-x}$$

- Make a valid guess for the particular solution.
- Find the general solution

Key:

- The differential equation is non-homogeneous with constant coefficients.

Let's find the homogeneous part of the solution to determine if we need to multiply our guess by x .

Characteristic equation:

$$r^2 + 2r + 1 = 0$$

$r = -1$ is a repeated root of the characteristic equation.

The general solution for the corresponding homogeneous equation is

$$y_h = c_1e^{-x} + c_2xe^{-x}$$

$g(x) = xe^{-x}$ is the product of a polynomial of degree 1 with an exponential function. The standard guess is

$$y_p = (Ax + B)e^{-x}$$

Since (-1) is a repeated root of the characteristic equation, or since Ae^{-x} and Bxe^{-x} are both solutions to the corresponding homogeneous equation, we need to multiply our guess by x^2 .

Valid guess: $y_p = (Ax + B)x^2e^{-x}$

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$$\begin{aligned} y_p &= (Ax^3 + Bx^2)e^{-x} \\ y'_p &= (3Ax^2 + 2Bx)e^{-x} - (Ax^3 + Bx^2)e^{-x} \\ &= (-Ax^3 + (3A - B)x^2 + 2Bx)e^{-x} \\ y''_p &= (-3Ax^2 + 2(3A - B)x + 2B)e^{-x} - (-Ax^3 + (3A - B)x^2 + 2Bx)e^{-x} \\ &= (Ax^3 + (-6A + B)x^2 + (6A - 4B)x + 2B)e^{-x} \end{aligned}$$

Plug in the differential equation:

$$\begin{aligned} (Ax^3 + (-6A + B)x^2 + (6A - 4B)x + 2B)e^{-x} + 2(-Ax^3 + (3A - B)x^2 + 2Bx)e^{-x} \\ + (Ax^3 + Bx^2)e^{-x} = xe^{-x} \\ ((A - 2A + A)x^3 + (-6A + B + 6A - 2B + B)x^2 + (6A - 4B + 4B)x + 2B)e^{-x} = xe^{-x} \\ ((6A)x + 2B)e^{-x} = xe^{-x} \end{aligned}$$

Identify the coefficients:

- Coefficient of xe^{-x} $6A = 1$, $A = \frac{1}{6}$.
- Coefficient of e^{-x} , $2B = 0$, $B = 0$.

The particular solution is

$$y_p = \frac{x^3}{6}e^{-x}$$

and the general solution is $y = c_1e^{-x} + c_2xe^{-x} + \frac{x^3}{6}e^{-x}$

Exercise 4. (5 points) Make a valid guess for a particular solution if the method of undetermined coefficient is to be used:

$$y'' + 4y = e^{-2x} + x^2 + \sin 3x + 5xe^{2x} + x^2 \cos 2x - 3xe^{2x} \sin 2x$$

Key: Let's find the solution to the corresponding homogeneous equation in order to determine which term(s) needs to be multiplied by x .

Characteristic equation:

$$r^2 + 4 = 0$$

$$r = \pm 2i$$

The general solution to the corresponding homogeneous equation is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

Let's make a guess for the particular solution:

$$\begin{aligned} y_p = & Ae^{-2x} + (Bx^2 + Cx + D) + E \sin(3x) + F \cos(3x) + (Gx + H)e^{2x} \\ & + (Ix^2 + Jx + K)x \cos(2x) + (Lx^2 + Mx + N)x \sin(2x) \\ & + (Px + Q)e^{2x} \sin(2x) + (Rx + S)e^{2x} \cos(2x) \end{aligned}$$

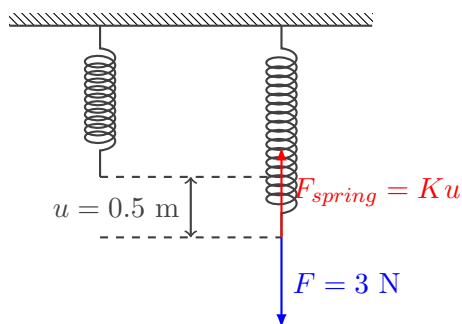
Exercise 5. (5 points) A spring with a mass of 2 kg has a damping constant 8 Ns/m (Newton * Second/Meter). A force of 3 N is required to stretch the spring 0.5 m beyond its natural length.

The spring is initially stretched 1 m beyond its natural length and released with no initial velocity.

- Find the position $u(t)$ as a function of the time.
- Find the mass that needs to be attached to the spring to produce critical damping.

Key:

- Let's find the spring constant:



When F is 3 N, and the spring is stretched 0.5 m, the spring is at an equilibrium. Therefore, $F + F_{spring} = 0$.

$$3 = 0.5K, \quad \implies \quad K = 6 \text{ N/m}$$

Let's find u

Newton Law:

$$mu'' = -ku - 8u', \quad u(0) = 1, u'(0) = 0$$

$$2u'' + 8u' + 6u = 0$$

$$u'' + 4u' + 3u = 0$$

Characteristic equation:

$$r^2 + 4r + 3 = 0$$

$$r = -3, \quad \text{or,} \quad r = -1$$

General solution:

$$u(t) = c_1e^{-3t} + c_2e^{-t}$$

Initial conditions:

$$u(0) = 1 = c_1 + c_2$$

$$u'(0) = 0 = -3c_1 - c_2$$

Adding the previous 2 equations,

$$\begin{aligned} -2c_1 &= 1 \\ c_1 &= -\frac{1}{2}, \quad c_2 = \frac{3}{2} \end{aligned}$$

the position $u(t)$ is $\boxed{u(t) = \frac{-e^{-3t}}{2} + \frac{3e^{-t}}{2}}$

- Newton law:

$$mu'' + 8u' + 6u = 0$$

The system is critically damped if the characteristic equation has a repeated root, i.e. if the discriminant is 0.

Characteristic equation:

$$mr^2 + 8r + 6 = 0$$

discriminant = $b^2 - 4ac = 0$

$$8^2 - 24m = 0$$

The mass for a critically damped system is $\boxed{m = \frac{8}{3}kg}$