Q1 (10 points)

(a) Find the general solution to $y'' + 2y' + y = 0$.

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r = -1$

$\boxed{y(t) = C_1 e^{-t} + C_2 t e^{-t}}$

(b) Write down a linear, homogenenous 2nd order differential equation with constant coefficients that has $y(t) = C_1 \cos 4t + C_2 \sin 4t$ as its general solution.

$\omega = 4$

$\boxed{y'' + 16y = 0}$

(c) Find the **real part** of the complex function $f(t) = e^{(2+3i)t}$.

$e^{2t} e^{3it} = e^{2t} \cos 3t + i e^{2t} \sin 3t$

real part: $\boxed{e^{2t} \cos 3t}$

Q2 (10 points)

Solve the initial value problem

$$y'' - y = \cos 2t, \qquad y(0) = 1, \quad y'(0) = 0.$$

1) Find y_h $\quad y'' - y = 0$

$r^2 - 1 = 0$

$(r+1)(r-1) = 0$

$r = 1, -1$

$y_h = C_1 e^t + C_2 e^{-t}$

2) Find y_p. Guess $y_p = A\cos 2t$

$y_p'' = -4A\cos 2t$

$-4A\cos 2t - A\cos 2t = \cos 2t, \quad -5A = 1, \quad A = -\frac{1}{5}$

$y_p = -\frac{1}{5}\cos 2t$

3) Solve IVP. $\quad y = C_1 e^t + C_2 e^{-t} - \frac{1}{5}\cos 2t$

$y(0) = 1 \Rightarrow 1 = C_1 + C_2 - \frac{1}{5}, \quad C_1 + C_2 = \frac{6}{5}$

$y'(t) = C_1 e^t - C_2 e^{-t} + \frac{2}{5}\sin 2t$

$y'(0) = 0 \Rightarrow 0 = C_1 - C_2, \quad C_1 = C_2$

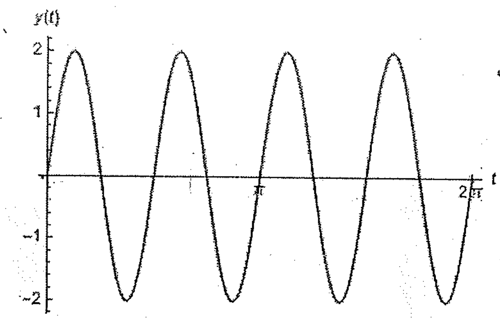
$2C_1 = \frac{6}{5}, \quad C_1 = \frac{3}{5}$

$C_2 = \frac{3}{5}$

$$\boxed{y(t) = \frac{3}{5}e^t + \frac{3}{5}e^{-t} - \frac{1}{5}\cos 2t}$$

Q3 (10 points)

(a) A 1 kg mass is attached to a spring. When pushed, the following motion is observed (here $y(t)$ indicates displacement in meters):



$A = 2$

$\omega = 4$

$\omega = \sqrt{\frac{k}{m}}$

$4 = \sqrt{\frac{k}{1}}$, $k = 16$

(i) What is the spring constant k?

$\boxed{k = 16}$

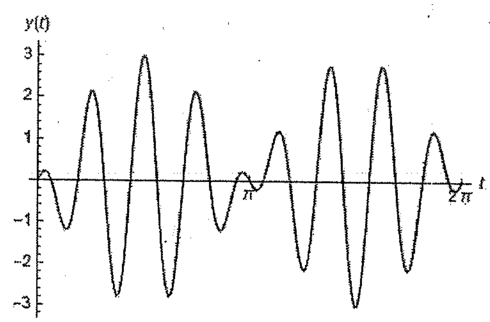
(ii) What is the (exact) initial velocity?

$y(t) = 2 \sin 4t$

$y'(t) = 8 \cos 4t$

$\boxed{y'(0) = 8}$

(b) A different mass-spring system is subjected to an oscillating external force, producing the following motion:



Write down a **possible** equation for $y(t)$ from the graph. You don't have to get it exactly right, and there is more than one possible answer.

$y(t) = \boxed{3 \sin t \sin 9t}$

Q4 (10 points)

(a) Solve the initial value problem.

$$y'' + \alpha^2 y = 1, \quad y(0) = 0, \quad y'(0) = 0,$$

where $\alpha > 0$ is a constant. Your solution $y(t)$ will involve the symbol α. (Hint: this is an inhomogeneous equation)

$y_h = C_1 \cos \alpha t + C_2 \sin \alpha t$

$y_p = A$

$y_p'' = 0$

$\alpha^2 A = 1 \Rightarrow A = \frac{1}{\alpha^2}$

$y(t) = C_1 \cos \alpha t + C_2 \sin \alpha t + \frac{1}{\alpha^2}$

$y(0) = 0 \Rightarrow 0 = C_1 + \frac{1}{\alpha^2}, \quad C_1 = -\frac{1}{\alpha^2}$

$y'(t) = -C_1 \alpha \sin \alpha t + C_2 \alpha \cos \alpha t$

$y'(0) = 0 \Rightarrow 0 = C_2 \alpha, \quad C_2 = 0$

$$\boxed{y(t) = -\frac{1}{\alpha^2} \cos \alpha t + \frac{1}{\alpha^2}}$$

(b) Find $\lim_{\alpha \to 0} y(t)$, where $y(t)$ is your solution from part (a).

$\lim_{\alpha \to 0} \frac{1 - \cos \alpha t}{\alpha^2}$ idea: use L'Hospital's rule

$= \lim_{\alpha \to 0} \frac{t \sin \alpha t}{2\alpha}$

$= \lim_{\alpha \to 0} \frac{t^2 \cos \alpha t}{2} = \boxed{\frac{t^2}{2}}$

Q5 (10 points)

A block of mass m is attached to a spring with spring constant $k = 1$ kg/s^2 and damping coefficient $\gamma = 4$ kg/s.

(a) For what range of values of m will the system exhibit oscillations with decaying amplitude?

oscillations occur when $\gamma^2 - 4mk < 0$

$$16 - 4m < 0$$

$$4m > 16$$

$$\boxed{m > 4}$$

(b) Interestingly, there is one particular value of m that will cause the system to oscillate with a larger quasifrequency ω than any other value of m. Find this value of m.

$$my'' + 4y' + y = 0$$

$$mr^2 + 4r + 1 = 0, \quad r = \frac{-4 \pm \sqrt{16 - 4m}}{2m} = \frac{-4m}{2m} \pm i\frac{\sqrt{m-4}}{m}$$

this is the quasifrequency

We want to maximize $\frac{\sqrt{m-4}}{m}$ as a function of m.

Shortcut: we can maximize the square of $\frac{\sqrt{m-4}}{m}$ instead

Let $f(m) = \frac{m-4}{m^2}$.

Then $f'(m) = \frac{m^2 - 2m(m-4)}{m^4} = \frac{-m^2 + 8m}{m^4} = \frac{8-m}{m^3}$

$m = 8$ is a critical value. Since $f(m) = 0$ at $m = 4$, $f(8) > 0$, and $\lim_{m\to\infty} f(m) = 0$, it is indeed a maximum.

$$\boxed{m = 8 \text{ kg}}$$