Name: ______________________________ Student number: __________________

Mathematics 207 J
University of Washington

February 23, 2022

MIDTERM 2 SOLUTIONS

Here are the rules:

- This exam is closed book. No note sheets, calculators, or electronic devices are allowed.
- In order to receive credit, you must **show all of your work**; to obtain full credit, you must provide mathematical justifications. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Give numerical answers in exact form (for example $\ln(\frac{\pi}{3})$ or $5\sqrt{3}$ or $e^{2.5}$).
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 5 pages, plus a cover sheet. Please make sure that your exam is complete.

$$\cos(\alpha-\beta) = \cos\alpha\,\cos\beta + \sin\alpha\,\sin\beta$$

$$\cos\alpha-\cos\beta = -2\sin\frac{\alpha+\beta}{2}\,\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha+\cos\beta = 2\cos\frac{\alpha+\beta}{2}\,\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha-\sin\beta = 2\cos\frac{\alpha+\beta}{2}\,\sin\frac{\alpha-\beta}{2}$$

$$\sin\alpha+\sin\beta = 2\sin\frac{\alpha+\beta}{2}\,\cos\frac{\alpha-\beta}{2}$$

Problem	Possible	Score
1	12	
2	10	
3	7	
4	14	
5	12	
Total	55	

Problem 1. (12 points)

(a) (9 points) Solve the following initial value problem.

$$y'' + 2y' + 5y \;=\; 5\cos t, \qquad y(0) = 1, \quad y'(0) = 0.$$

(b) (3 points) Identify the transient and the steady state components of your answer.

Solution. $r^2 + 2r + 5 = 0 \;\Rightarrow\; r = -1 \pm 2i$, homogeneous general solution:

$$y(t) = C_1 e^{-t}\cos(2t) + C_2\, e^{-t}\sin(2t)$$

Undetermined coefficients: plug in $Y(t) = A\cos(t) + B\sin(t)$, then

$$Y'' + 2Y' + 5Y = (4A + 2B)\cos(t) + (4B - 2A)\sin(t),$$

set equal to $5\cos(t)$ and solve $A = 1$, $B = \frac{1}{2}$.
General solution to the inhomogeneous equation:

$$y(t) = C_1 e^{-t}\cos(2t) + C_2\, e^{-t}\sin(2t) + \cos(t) + \tfrac{1}{2}\sin(t)$$

Set $y(0) = 1$ to find $C_1 = 0$, and $y'(0) = 0$ to find $C_2 = -\frac{1}{4}$, so

$$\boxed{y(t) = -\tfrac{1}{4}\, e^{-t}\sin(2t) + \cos(t) + \tfrac{1}{2}\sin(t)}$$

Transient part: $\boxed{-\frac{1}{4}\, e^{-t}\sin(2t)}$ Steady state: $\boxed{\cos(t) + \frac{1}{2}\sin(t)}$

Problem 2. (10 points) Solve the following initial value problem:

$$y'' + 25y = \sin(5t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution. $r^2 + 25 = 0 \;\Rightarrow\; r = \pm 5i$, so homogeneous general solution:

$$y(t) = C_1 \cos(5t) + C_2 \sin(5t)$$

Driving force is part of homogeneous solution, so for undetermined coefficients try

$$Y(t) = At\cos(5t) + Bt\sin(5t)$$

Then

$$Y'' + 25Y = -10A\sin(5t) + 10B\cos(5t)$$

Set equal to $\sin(5t)$ to get $A = -\frac{1}{10}$, $B = 0$. So general solution to inhomogeneous equation is:

$$y(t) = C_1 \cos(5t) + C_2 \sin(5t) - \tfrac{1}{10}\, t\cos(5t)$$

Set $y(0) = 0$ to find $C_1 = 0$, and $y'(0) = 0$ to find $C_2 = \frac{1}{50}$. Final answer:

$$\boxed{y(t) = \tfrac{1}{50}\sin(5t) - \tfrac{1}{10}\, t\cos(5t)}$$

Problem 3. (7 points) A spring is observed to stretch $\frac{1}{2}$ meter when a force of 3 newtons is applied to it. A viscous damper is observed to yield a resistance of 2 newtons when it is moved at a velocity of 1 meter/second.

A mass of 2 kg is hung from the spring and attached to the viscous damper. It is then pulled $\frac{1}{2}$ meter below its rest position and released with 0 initial velocity.

Write down the differential equation and initial conditions for $u(t)$, the position of the mass at time t relative to its rest position, where $u > 0$ means the mass is above the rest position. **Do not solve the equation**. **(And yes, this problem is really short.)**

Solution. In meter-kilogram-seconds units, $m = 2$, $k = 3/(1/2) = 6$, $\gamma = 2$, so

$$\boxed{2u'' + 2u' + 6u = 0, \quad u(0) = -\tfrac{1}{2}, \quad u'(0) = 0}$$

Problem 4. (14 points) Consider the initial value problem

$$u'' + 2u' + \tfrac{5}{4}u = 0\,, \qquad u(0) = 2\,, \quad u'(0) = 1.$$

(a) (5 points) Solve the initial value problem.

$$u(t) = e^{-t}\Big(2\cos(\tfrac{1}{2}t) + 6\sin(\tfrac{1}{2}t)\Big)$$

(b) (6 points) Express the answer in the form $u(t) = A\,e^{\rho t}\cos(\omega t - \phi)$, where $A > 0$.

$A = \sqrt{40}$ $\qquad \rho = -1$ $\qquad \omega = \frac{1}{2}$ $\qquad \phi = \tan^{-1}(3)$

(c) (3 points) Find the first time $t > 0$ at which $u(t) = 0$.

$$t = 2\tan^{-1}(3) + \pi$$

Solution. (a). $r^2 + 2r + \frac{5}{4} = 0 \;\Rightarrow r = -1 \pm \frac{1}{2}i$, homogeneous general solution:

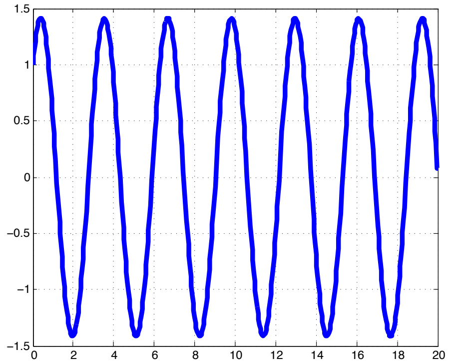
$$u(t) = C_1 e^{-t}\cos(\tfrac{1}{2}t) + C_2\, e^{-t}\sin(\tfrac{1}{2}t)$$

Set $u(0) = 2$ to get $C_1 = 2$ and $u'(0) = 1$ to get $C_2 = 6$.

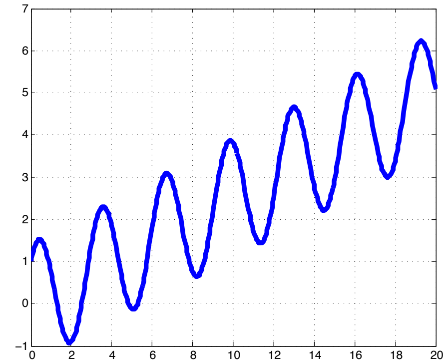
(b). Need write $(2,6) = \big(A\cos(\phi), A\sin(\phi)\big)$. $A = \sqrt{2^2 + 6^2}$, and since the point $(2,6)$ is in the first quadrant, $\phi = \tan^{-1}(3)$.

(c). Need find the first $t > 0$ so that $\cos(\frac{1}{2}t - \tan^{-1}(3)) = 0$. Since $0 < \tan^{-1}(3) < \frac{\pi}{2}$, this will occur when $\frac{1}{2}t - \tan^{-1}(3) = \frac{\pi}{2}$.

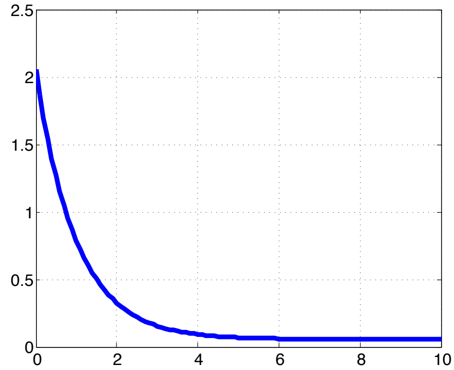
Problem 5. (12 points) Each of the 6 differential equations below has a solution that is plotted in one of the graphs. Match each of the differential equations to its solution. (Note: only 6 of the graphs will correspond to an equation.)



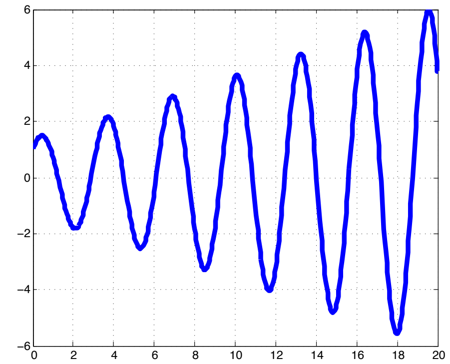
(a)



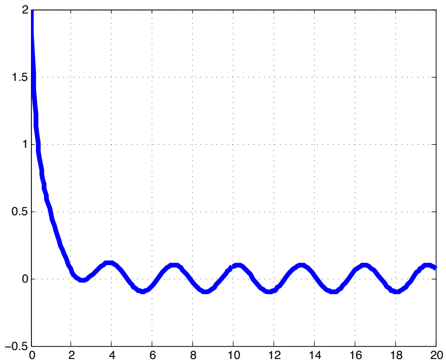
(b)



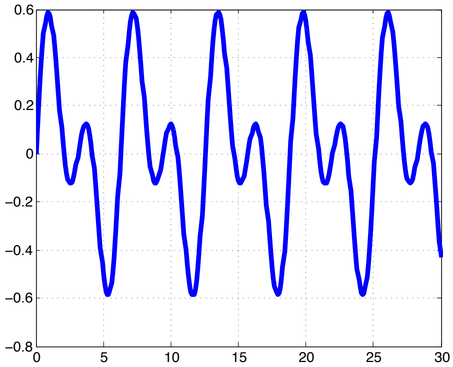
(c)



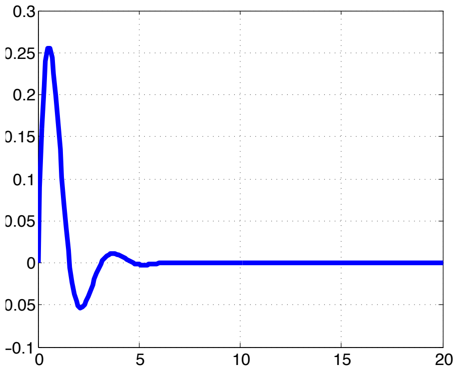
(d)



(e)



(f)



(g)

Differential Equation	Graph
$y'' + 4y = \sin(t)$	f
$y'' + 4y = \cos(2t)$	d
$y'' + 4y = 0$	a
$y'' + 5y' + 4y = 0$	c
$y'' + 5y' + 4y = \cos(2t)$	e
$y'' + 2y' + 5y = 0$	g