Math 307C, Autumn 2017 **Name:** ______________________________

Test #2 (100 points total)

Instructions: Please read the following instructions before you start the test:

- Show all work. Answers without sufficient work may not get full credit. Give all answers in exact form. Calculators are not allowed.
- Be sure to either erase or scratch out any work that you do not want graded—two answers for a problem will nullify your solution, even if one of them is correct.
- If you need additional space, you may use backs of pages.
- There is a table of trig formulas and integrals on the last page.
- You are allowed one 11" by 8" sheet of handwritten notes. Writing on both sides of the paper is allowed.

1.	(60 points)	
2.	(30 points)	
3.	(10 points)	
TOTAL	(100 points)	

(1) **(60 points)** Solve each of the following initial value problems.

(a) **(15 points)** $y'' - 4y = 0$, $y(0) = 1$, $y'(0) = 0$.

Seek $y = e^{rt}$

$r^2 - 4 = 0$

$r = \pm 2$

$y(t) = c_1 e^{2t} + c_2 e^{-2t}$

$1 = y(0) = c_1 + c_2$

$0 = y'(0) = 2c_1 - 2c_2$

$1 = c_1 + c_2$

$0 = c_1 - c_2$

$1 = 2c_1 \Rightarrow c_1 = 1/2$

$1 = 2c_2 \Rightarrow c_2 = 1/2$

$$\boxed{y(t) = \tfrac{1}{2} e^{2t} + \tfrac{1}{2} e^{-2t}}$$

(b) **(15 points)** $y'' - 4y = e^{2t}$, , $y(0) = 1$, $y'(0) = 0$.

Seek $y = y_p + y_h$

$y_h = c_1 e^{2t} + c_2 e^{-2t}$

$y_p = A e^{2t}$

Does y_p appear in y_h?

Yes! $y_p = Ate^{2t}$

Find A

$y_p = Ate^{2t}$

$y_p' = A(2te^{2t} + e^{2t})$

$y_p'' = A(4te^{2t} + 4e^{2t})$

$e^{2t} = y_p'' - 4y_p = A \cdot 4e^{2t}$

so $A = 1/4$

$y(t) = \frac{te^{2t}}{4} + c_1 e^{2t} + c_2 e^{-2t}$

$1 = y(0) = c_1 + c_2$

$0 = y'(0) = \frac{1}{4} + 2c_1 - 2c_2$

$1 = c_1 + c_2$

$-\frac{1}{8} = c_1 - c_2$

$\frac{7}{8} = 2c_1 \Rightarrow c_1 = \frac{7}{16}$

$c_2 = \frac{9}{16}$

$$\boxed{y(t) = \frac{te^{2t}}{4} + \frac{7}{16} e^{2t} + \frac{9}{16} e^{-2t}}$$

(c) (15 points) $2y'' + 4y' + 10y = 0$, $y(0) = 1$, $y'(0) = 1$.

$2r^2 + 4r + 10 = 0$
$r^2 + 2r + 5 = 0$
$(r+1)^2 + 4 = 0$
$(r+1)^2 = -4$
$r = -1 \pm 2i$

$y(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$

$1 = y(0) = c_1 + 0$
$1 = y'(0) = -c_1 + 2c_2$

$2 = 2c_2 \Rightarrow c_2 = 1$
$c_1 = 1$

$y(t) = e^{-t}\cos 2t + e^{-t}\sin 2t$

(d) (15 points) $\cancel{2}y'' + \overset{2}{\cancel{4}}y' + \overset{5}{\cancel{10}}y = \overset{6}{\cancel{12}}\cos(3t)$, $y(0) = 0$, $y'(0) = 0$.

$y_h = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$

Seek $y_p = A\cos 3t + B\sin 3t$
Does y_p appear in y_h? No

$y_p = A\cos 3t + B\sin 3t$
$y_p' = 3B\cos 3t - 3A\sin 3t$
$y_p'' = -9A\cos 3t - 9B\sin 3t$

$6\cos 3t = y_p'' + 2y_p' + 5y_p = (-4A + 6B)\cos 3t + (-4B - 6A)\sin 3t$

$\left.\begin{array}{l} 6 = -4A + 6B \\ 0 = -6A - 4B \end{array}\right\} \Rightarrow \left.\begin{array}{l} 3 = -2A + 3B \\ 0 = -3A - 2B \end{array}\right\} \Rightarrow \begin{array}{l} 3 = -2A + 3B \\ 0 = -\frac{9}{2}A - 3B \end{array}$

$3 = -\frac{13}{2}A \Rightarrow A = \frac{-6}{13}$
$B = \frac{9}{13}$

$y = -\frac{6}{13}\cos 3t + \frac{9}{13}\sin 3t + c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$

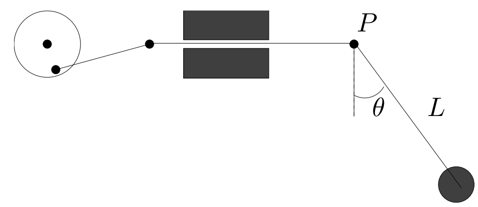
$0 = y(0) = \frac{-6}{13} + c_1 \Rightarrow c_1 = \frac{6}{13}$

$0 = y'(0) = \frac{27}{13} - \frac{6}{13} + 2c_2 \Rightarrow c_2 = \frac{21}{26}$

$y = -\frac{6}{13}\cos 3t + \frac{9}{13}\sin 3t + \frac{6}{13}e^{-t}\cos 2t + \frac{21}{26}e^{-t}\sin 2t$

(2) **(30 points)** A simple pendulum of mass m and length L is hinged at a point P (see figure). If the wheel at the left of the figure rotates at a rate of ω radians/second it forces the point P to move periodically back and forth.

For small angle θ (where $\sin(\theta) \approx \theta$) the angle θ satisfies the differential equation



$$mL\frac{d^2\theta}{dt^2} + mg\theta = A\sin(\omega t).$$

Assume for simplicity that
$m = 1$ kg, $L = 10$ meter, $A = 20$ Newtons,
$g = 10$ meter/sec^2, and $\omega = 1$ rad/sec.

Find the solution satisfying the initial conditions $\theta(0) = \dot{\theta}(0) = 0$.

Be sure to clearly show the general solution to the ODE and how you got it.

$$10\ddot{\Theta} + 10\Theta = 20\sin t$$

$$\boxed{\begin{array}{c}\ddot{\Theta} + \Theta = 2\sin t \\ \Theta(0) = \dot{\Theta}(0) = 0\end{array}}$$

$$\Theta_h = c_1\cos t + c_2\sin t$$

$$\Theta_p = At\cos t + Bt\sin t$$

$$\Theta_p = At\cos t + Bt\sin t$$

$$\dot{\Theta}_p = (A+Bt)\cos t + (B-At)\sin t$$

$$\ddot{\Theta}_p = (2B-At)\cos t - (2A+Bt)\sin t$$

$$2\sin t = \ddot{\Theta}_p + \Theta_p = 2B\cos t - 2A\sin t$$

$$A = -1 \;;\; B = 0$$

$$\Theta(t) = -t\cos t + c_1\cos t + c_2\sin t$$

$$0 = \Theta(0) = c_1 \Rightarrow c_1 = 0$$

$$0 = \dot{\Theta}(0) = -1 + c_2 \Rightarrow c_2 = 1$$

$$\boxed{\Theta(t) = -t\cos t + \sin t}$$

(3) **(10 points)** Suppose that you are designing a new shock absorber for a small automobile. The automobile has a mass of 800 kilograms and the combined effect of the springs in the suspension system is that of a spring constant of 16000[redacted] Newtons/meter.

(a) **(5 points)** Before a damping mechanism is installed in the automobile, when it hits a bump it will bounce up and down. What is the period of the oscillations when it hits a bump?

(b) **(5 points)** Your job is to design a damping mechanism which eliminates oscillations when the automobile hits a bump. What is the minimum value of the effective damping constant that you need?

(a) $8\cancel{0}\cancel{0}\ddot{y} + 160\cancel{0}\cancel{0}y = 0$

$$\ddot{y} + 20y = 0$$

$$\omega_0 = \sqrt{20}$$

$$\boxed{P_0 = \frac{2\pi}{\sqrt{20}}}$$

(b) $\ddot{y} + \gamma\dot{y} + 20y = 0$

$$r^2 + \gamma r + 20 = 0$$

$$\left(r + \frac{\gamma}{2}\right)^2 = \frac{\gamma^2}{4} - 20$$

Real solutions if $\frac{\gamma^2}{4} > 20$ i.e. $\gamma^2 > 80$

$\gamma \geq 4\sqrt{5}$

Damping Coeff $= 800\gamma$

so DampinCoeff $\geq 3200 \cdot \sqrt{5}$ kg/sec

Selected Formulas

Trigonometry

$$\sin x = \cos\left(\frac{\pi}{2} - x\right); \ \cos x = \sin\left(\frac{\pi}{2} - x\right);$$

$$\sin(x+\pi) = -\sin x; \ \cos(x+\pi) = -\cos x.$$

$$\sin^2 x + \cos^2 x = 1; \ \tan^2 x + 1 = \sec^2 x; \ 1 + \cot^2 x = \csc^2 x;$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \, \tan y}$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2}; \ \cos^2 x = \frac{1+\cos(2x)}{2}; \ 2\sin(x)\cos(x) = \sin(2x)$$

$$\cos x - \cos y = 2 \, \sin\left(\frac{x+y}{2}\right) \, \sin\left(\frac{y-x}{2}\right)$$

$$\sin x - \sin y = 2 \, \cos\left(\frac{x+y}{2}\right) \, \sin\left(\frac{x-y}{2}\right)$$

Calculus

Integration by parts: $\int u\,dv = uv - \int v\,du$;

Standard integrals:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1); \ \int \frac{dx}{x} = \ln|x| + C; \ \int e^x\,dx = e^x + C;$$

$$\int \sin x\,dx = -\cos x + C; \ \int \cos x\,dx = \sin x + C; \ \int \tan x\,dx = -\ln|\cos x| + C;$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C; \ \int \frac{xdx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + C;$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C; \ \int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C;$$

$$\int \frac{x\,dx}{1+x^2} = \frac{1}{2}\ln(1+x^2) + C$$