Q1 (10 points)

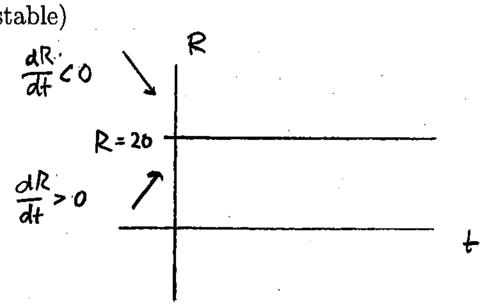
Radon, a radioactive element, commonly enters into basements through cracks in the floor or walls.

Let $R(t)$ denote the amount of radon gas (measured in pCi, or *picocuries*) in a basement room at any given time t (measured in days). You are given that $R(t)$ satisfies the differential equation

$$\frac{dR}{dt} = -0.2R + 4$$

(a) This equation has an equilibrium solution. **What kind** of equilibrium is it? (stable, unstable, semistable)

stable



(b) Initially there is no radon gas present. Find the exact amount of radon present after 3 days.

$$\frac{dR}{dt} = -0.2(R-20)$$

$$\frac{1}{R-20}\,dR = -0.2\,dt$$

$$\ln|R-20| = -0.2t + C$$

$$|R-20| = e^{-0.2t}e^{C}$$

$$R-20 = Ae^{-0.2t}$$

$$R(t) = Ae^{-0.2t} + 20$$

$$R(0) = 0 \Rightarrow A = -20$$

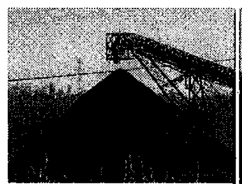
$$R(t) = 20 - 20e^{-0.2t}$$

$$R(3) = \boxed{20 - 20e^{-0.6}}$$

(about 9 pCi)

Q2 (10 points)

Coal is being dumped onto a pile at a constant rate of 3 m^3 per minute. Assume the pile is shaped like a cone with the radius of its base equal to its height, and assume that it retains the shape of a cone (with base radius always equal to its height) as it increases in size.



(a) Write a differential equation for $\frac{dV}{dt}$, where V is the volume of coal in the pile in m^3 and t is time measured in minutes.

$$\boxed{\frac{dV}{dt} = 3}$$

(b) Write a differential equation for $\frac{dh}{dt}$, where h is the height of the pile. (Hint: the formula for volume of a cone is $V = \frac{1}{3}\pi r^2 h$, where r is the base radius and h is the height)

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^3 \quad \text{(note: } r \text{ isn't constant)}$$

$$\frac{dV}{dt} = \pi h^2 \frac{dh}{dt}$$

$$\boxed{\frac{dh}{dt} = \frac{3}{\pi h^2}}$$

(c) How would your differential equation in part (b) change if time were measured in hours instead?

$$\boxed{\frac{dh}{dt} = \frac{180}{\pi h^2}}$$

Q3 (10 points)

Find an explicit solution $y(t)$ to the initial value problem:

$$\frac{dy}{dt} = y + \frac{1}{y} \qquad y(0) = -2$$

$$\frac{dy}{dt} = \frac{y^2+1}{y}$$

$$\int \frac{y}{y^2+1}\, dy = \int dt$$

$$\frac{1}{2} \ln|y^2+1| = t + C \qquad \text{(u-substitution)}$$

$$\ln|y^2+1| = 2t + C$$

$$|y^2+1| = e^{2t} e^{C}$$

$$y^2 + 1 = Ae^{2t}$$

$$y^2 = Ae^{2t} - 1$$

$$y = \pm\sqrt{Ae^{2t}-1}$$

$\underline{y(0) = -2}$

$$-2 = \pm\sqrt{A-1}$$

$A = 5$, choose $-$

$$\boxed{y(t) = -\sqrt{5e^{2t}-1}}$$

Q4 (10 points)

Find the general solution to

$$\frac{dy}{dx} = 1 + y + x^2$$

$$\frac{dy}{dx} - y = 1 + x^2 \qquad \mu(x) = e^{-x}$$

$$\frac{d}{dx}(e^{-x}y) = e^{-x} + x^2 e^{-x}$$

$$e^{-x}y = \underbrace{\int e^{-x}dx}_{-e^{-x}} + \underbrace{\int x^2 e^{-x}dx}_{-x^2e^{-x} - 2xe^{-x} - 2e^{-x}}$$

$$e^{-x}y = -x^2e^{-x} - 2xe^{-x} - 3e^{-x} + C$$

$$\boxed{y(x) = -x^2 - 2x - 3 + Ce^{x}}$$

$u = x^2$ $\quad v = -e^{-x}$

$du = 2x\,dx$ $\quad dv = e^{-x}dx$

$$\int x^2 e^{-x}dx = -x^2e^{-x} + 2\int xe^{-x}dx$$

$u = x$ $\quad v = -e^{-x}$

$du = dx$ $\quad dv = e^{-x}dx$

$$= -x^2e^{-x} + 2\left(-xe^{-x} + \int e^{-x}dx\right)$$

$$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x}$$

Q5 (10 points)

A large room is shaped like a rectangular prism with width 5 m, length 5 m, and height 4 m. $V = 100\ m^3$

A vent begins blowing air containing carbon monoxide at a concentration of α mg/m^3 into the room at a rate of 2 m^3 per minute. Another vent **removes** the well-mixed air from the room at the same rate.

Initially there was no carbon monoxide in the air of the room, but 50 minutes later an alarm goes off, indicating that the carbon monoxide concentration in the room has exceeded a level of 100 mg/m^3.

Determine the value of α.

Let y = mass of CO in room (in mg)

$$\frac{dy}{dt} = 2\alpha - 2 \cdot \frac{y}{100}$$

$$\int \frac{1}{\alpha - \frac{y}{100}} dy = \int 2\, dt$$

$$-100 \ln\left|\alpha - \frac{y}{100}\right| = 2t + C$$

$$\ln\left|\alpha - \frac{y}{100}\right| = -\frac{1}{50}t + C$$

$$\alpha - \frac{y}{100} = Ae^{-\frac{1}{50}t}$$

$$\frac{y}{100} = \alpha - Ae^{-\frac{1}{50}t}$$

$$y = 100\alpha - Ae^{-\frac{1}{50}t}$$

$$y(0) = 0 \Rightarrow A = 100\alpha$$

$$y = 100\alpha - 100\alpha e^{-\frac{1}{50}t}$$

$$y(t) = 100\alpha\left(1 - e^{-\frac{1}{50}t}\right)$$

$$y(50) = 100 \cdot 100 = 10000$$

$$10000 = 100\alpha\left(1 - e^{-1}\right)$$

$$\boxed{\alpha = \frac{100}{1 - e^{-1}}}$$