Name: ______________________________

Mathematics 207 J
University of Washington

February 2, 2022

MIDTERM 1 SOLUTIONS

Here are the rules:

- This exam is closed book. No note sheets, calculators, or electronic devices are allowed.
- In order to receive credit, you must **show all of your work**; to obtain full credit, you must provide mathematical justifications. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Give numerical answers in exact form (for example $\ln(\frac{\pi}{3})$ or $5\sqrt{3}$ or $e^{2.5}$).
- Simplify $e^{a\ln(x)} = x^a$ for $x > 0$.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 6 pages, plus a cover sheet. Please make sure that your exam is complete.
- You have 50 minutes to complete the exam.
- HAVE FUN!

Problem	Possible	Score
1	10	
2	8	
3	10	
4	11	
5	16	
Total	55	

Good Luck!

Problem 1. Consider the initial value problem,

$$\frac{dy}{dt} + \frac{2y}{5+t} = 6, \qquad y(0) = 0.$$

(a) (2 points) Circle your answer

(i) Is this a *linear* differential equation? **[YES]** **NO**

(ii) Is this a *separable* differential equation? **YES** **[NO]**

(b) (8 points) Solve the initial value problem.

Solution. Integrating factor solves

$$\frac{dm}{dt} = \frac{2m}{5+t} \Rightarrow \frac{dm}{m} = \frac{2dt}{5+t} \Rightarrow \ln(m) = 2\ln(5+t)$$

so

$$\mu = e^{2\ln(5+t)} = (5+t)^2$$

Equation becomes

$$(5+t)^2 \frac{dy}{dt} + 2(5+t)y = 6(5+t)^2$$

which we rewrite as

$$\frac{d}{dt}\Big((5+t)^2 y\Big) = 6(5+t)^2$$

Integrate to get

$$(5+t)^2 y = 2(5+t)^3 + C$$

Set $t = 0$ and $y = 0$ to solve

$$C = -250$$

Solve for y to get

$$\boxed{y = 2(5+t) - 250(5+t)^{-2}}$$

Problem 2. (8 points) You deposit \$10000 into a savings account at 5% annual interest, compounded continuously. You withdraw money from the account at a continuous rate of \$1000 per year. After how many years will the account balance be \$0?

Solution. Equation is

$$\frac{dP}{dt} = 0.05\,P - 1000$$

Separation of variables method (can also use linear equation method):

$$\frac{dP}{dt} = 0.05(P - 20000)$$

$$\frac{dP}{P - 20000} = 0.05\,dt \quad \Rightarrow \quad \ln|P - 20000| = 0.05\,t + C$$

$$\Rightarrow \quad |P - 20000| = e^C\,e^{0.05t} \quad \Rightarrow \quad P - 20000 = \pm e^C\,e^{0.05t}$$

Replace $\pm e^C$ by C to finally write

$$P(t) = 20000 + C\,e^{0.05t}.$$

Set $t = 0$ and $P = 10000$ to find $C = -10000$,

$$P(t) = 20000 - 10000\,e^{0.05t}.$$

Find t so that $P(t) = 0$ gives

$$e^{0.05t} = 2 \quad \Rightarrow \quad t = 20\ln(2).$$

Problem 3. (10 points) Consider the initial value problem

$$t^2 \frac{dy}{dt} = \frac{1}{y+3}, \qquad y(1) = -5.$$

Solve the initial value problem. Give an explicit formula for y.

Solution. Separation of variables:

$$(y+3)\, dy = \frac{dt}{t^2}$$

Integrate

$$\frac{1}{2}(y+3)^2 = -\frac{1}{t} + C$$

Set $t = 1$ and $y = -5$ to find $C = 3$.

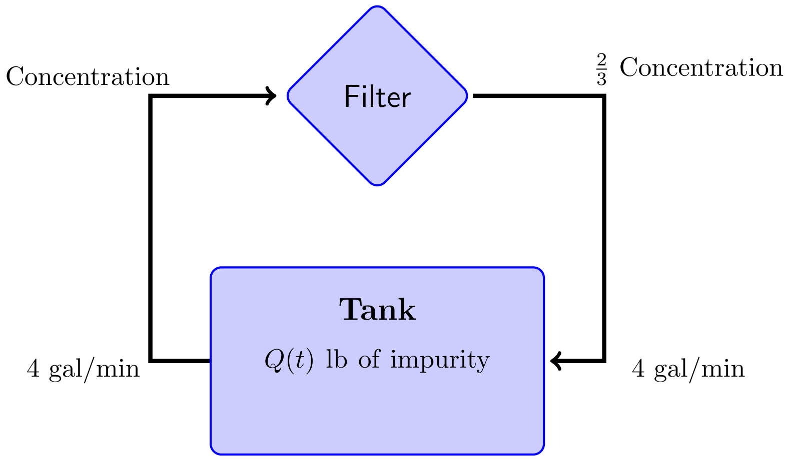
Solve (be careful about $\pm$ in this step!)

$$y + 3 = \pm\sqrt{-\frac{2}{t} + 6}$$

Since $y(1) = -5$ we need to take the $-$ sign, leading to

$$y = -3 - \sqrt{-\frac{2}{t} + 6}$$

Problem 4. A tank holding water that contains an impurity Q is attached to a recirculating filter, as pictured below. The liquid passes through the filter at the rate of 4 gal/min. The filter removes 1/3 of the amount of Q that passes through it, and lets the remaining 2/3 go back into the tank.



(a) (5 points) The tank contains 20 gallons of water. Initially there are 2 pounds of Q dissolved in the water. Pose a differential equation with initial value for the amount $Q(t)$ in the tank at time t.

Solution. The rate that Q flows through the filter is $4Q/20 = Q/5$. The filter removes 1/3 of this,

$$\boxed{\frac{dQ}{dt} = -\frac{Q}{15}} \qquad \boxed{Q(0)=2}$$

(b) (6 points) Now suppose that the tank initially contains 2 pounds of Q dissolved in 20 gallons of water. Water containing 1 pound of Q per gallon is added to the tank at the rate of 2 gallons per minute. The volume of water in the tank therefore increases. The filter continues to operate as above at 4 gallons per minute, removing 1/3 of the amount of Q passing through it. Pose a differential equation with initial value for the amount $Q(t)$ in the tank at time t.

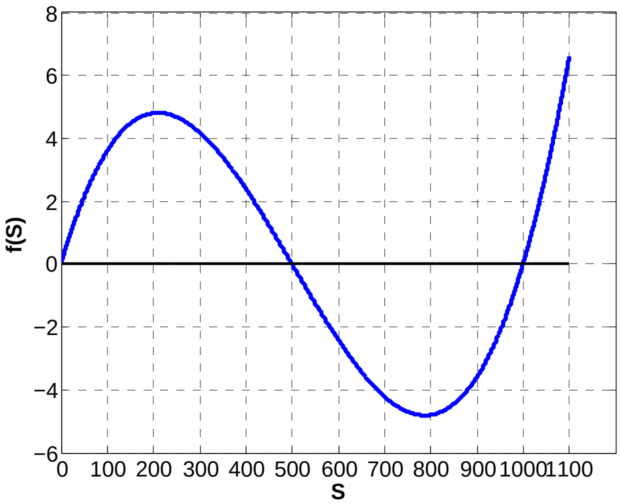
Solution. The volume increases 2 gal/min, so $V = 20 + 2t$. Also, salt is added to the tank at the rate of 2 pounds/min. The rate that Q flows through the filter is now $4Q/(20 + 2t)$. The filter removes 1/3 of this,

$$\boxed{\frac{dQ}{dt} = 2 - \frac{4}{3}\frac{Q}{20+2t}} \qquad \boxed{Q(0)=2}$$

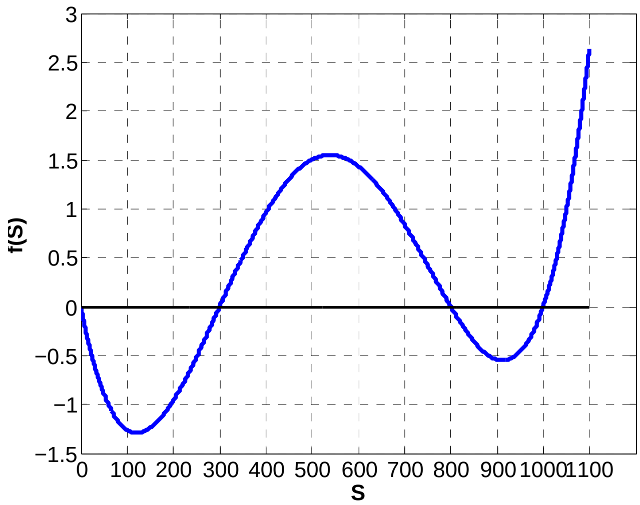
Problem 5. Biologists have observed that a population of wild pigs satisfies the following differential equation, where t is in days:

$$\frac{dS}{dt} = f(S).$$

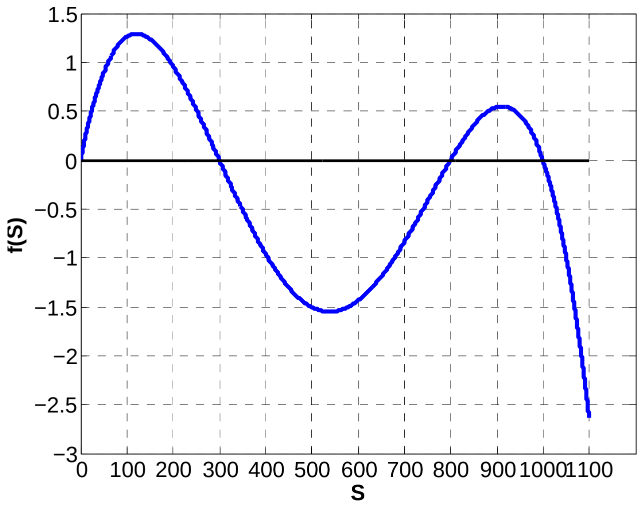
Four proposed expressions for $f(S)$ are shown below:



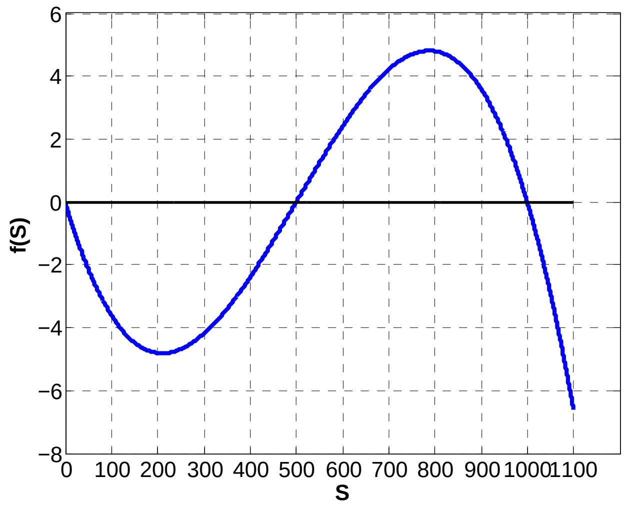
(a) Model A



(b) Model B



(c) Model C



(d) Model D

Part (a): (4 points) You observe

- If the population of pigs is 150, the population decreases until it reaches 0.
- If the population of pigs is 600, the population increases until it reaches 1000.

CIRCLE which model illustrates these two observations

Model A   Model B   Model C   Model D

There are more parts to this problem on the following page!

Answer parts (b) and (c) below based on the model you chose in part (a):

Part (b): (8 points) Determine the equilibrium solutions and classify each one as asymptotically stable or unstable. Sketch the direction field, and in your sketch draw an approximate graph of the solution $S(t)$ that satisfies $S(0) = 700$; be sure to indicate the behavior of $S(t)$ as t goes to $+\infty$.

Solution. For Model D, equilibrium solutions are $s = 0$, $s = 500$, and $s = 1000$. Both 0 and 1000 are stable, 500 is unstable.

For Model D, if $S(0) = 700$ then $S(t)$ increases to 1000 as $t \to \infty$.

Part (c): (4 points) The government plans to hunt a fixed number k of pigs per day. If there are 700 pigs when they start hunting, what value of k (approximately) will keep the population of pigs constant. EXPLAIN.

Solution. If $S = 700$ then from Model D, there are $f(700) \approx 4$ pigs born every day. Thus we need to hunt them at the rate of ≈ 4 per day to keep the population constant. Another way: the equation when we add in a hunting rate of k becomes

$$\frac{dS}{dt} = f(S) - k$$

$S = 700$ is an equilibrium solution to this equation when $f(700) - k = 0$, so again we obtain $k = f(700) \approx 4$.

Submitted by Name: ______________________________ *on February 2, 2022.*