Name (PRINT): ______________________

Student ID (PRINT): ______________________

Autumn 2022 – Introduction to Differential Equations
First Examination

Instructions

1. The use of all electronic devices is prohibited. Any electronic device needs to be turned off and placed in your bag. Any textbooks or notes also need to be placed in your bag.

2. Present your solutions in the space provided. Show all your work neatly and concisely. Clearly indicate your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

Scholastic dishonesty will not be tolerated and may result in terminating the midterm early. The work on this test is my own.

Signature:______________________

Exercise 1. (4 points) Choose the direction field corresponding to each differential equations. Justify your answers:

1. $y' = \frac{x}{y}$.
2. $y' = -xy$

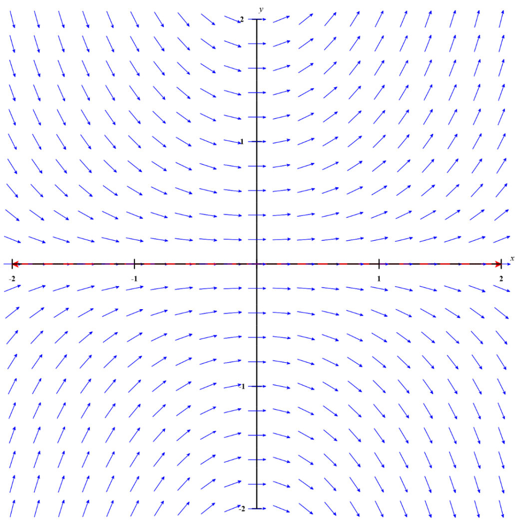
Key:

1. Properties of $y' = \frac{x}{y}$:
   - Horizontal tangents when $y' = 0$, i.e. when $x = 0$.
     The direction field has horizontal arrows when on the y-axis ($x = 0$).
     Candidates: Direction fields 1,2,3,4.
   - Arrows are pointing up if $y' = \frac{x}{y} > 0$. That happens when x and y have the same sign, in the first and third quadrant.
     Candidates: Direction fields 1,2, 6
   - When y gets close to 0, for a given x, $\frac{x}{y}$ goes to infinity. The tangent lines get close to being vertical.
     *Candidates:*Directions fields 2, 3
   - When $x = y$, the slope is 1.
     Candidates: Direction fields 2, 6

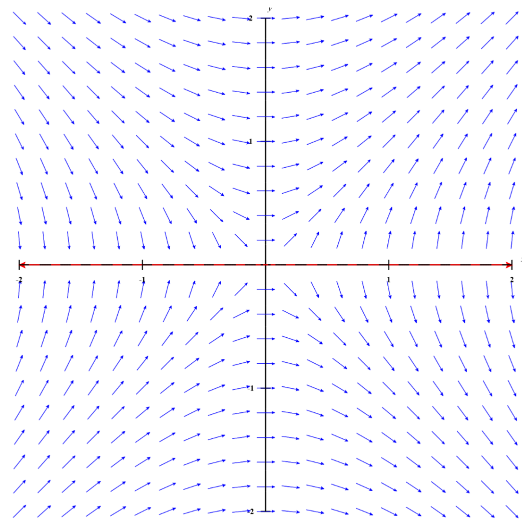
   In conclusion, the only direction fields that has all the properties is the direction field $\boxed{2}$
   .

2. Properties of $y' = -xy$:
   - Horizontal tangent when x or y is 0
     Candidates: Direction fields 1, 3
   - Positive slope when x and y have opposite signs.
     Candidates Direction fields 3, 4, 5

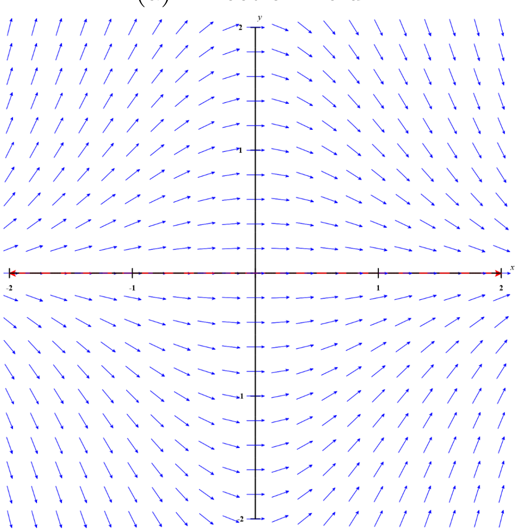
   The only direction field that satisfies both properties is $\boxed{3}$.



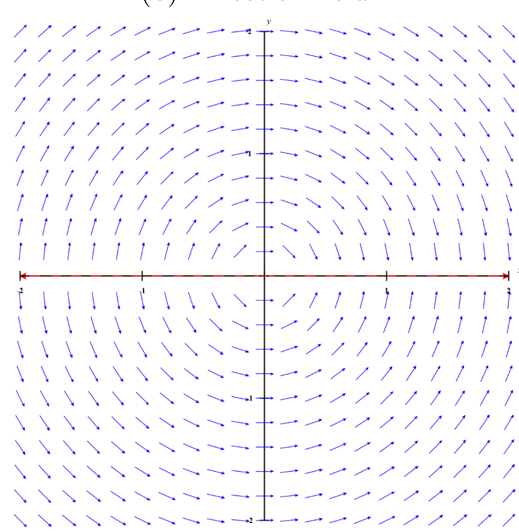
(a) Direction field 1



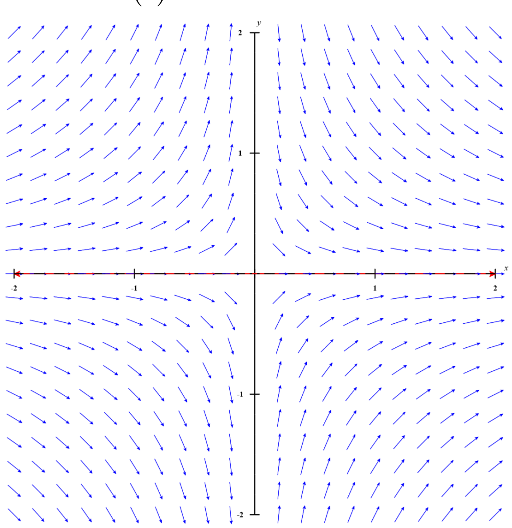
(b) Direction field 2



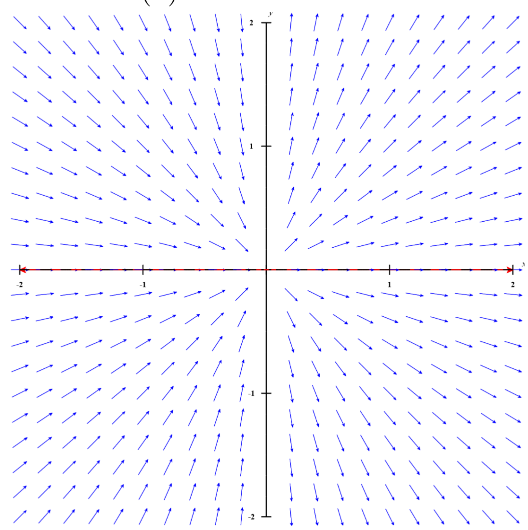
(c) Direction field 3



(d) Direction field 4



(e) Direction field 5



(f) Direction field 6

Exercise 2. (6 points) Find the general solutions. Find explicit solutions if possible.

1.
$$\frac{ty'}{t^4\cos(2t)+3y}=1$$

Key:
$$\frac{ty'}{t^4\cos(2t)+3y}=1$$

iff

$$ty' = t^4\cos 2t + 3y$$
$$ty' - 3y = t^4\cos 2t$$
$$y' - \frac{3}{t}y = t^3\cos 2t$$

The equation is linear. We can solve it using integrating factors.

- Integrating factor:
$$\mu(t) = e^{\int \frac{-3}{t}\,dt} = e^{-3\ln(t)} = t^{-3}$$
- Multiply the integrating factor:
$$\underbrace{\frac{y'}{t^3} - \frac{3}{t^4}y}_{\frac{d}{dt}\left(\frac{y}{t^3}\right)} = \cos 2t$$
- Integrate both sides:
$$\frac{y}{t^3} = \frac{\sin(2t)}{2} + C$$
- Divide by the integrating factor:
General solution $\boxed{y = \frac{t^3\sin(2t)}{2} + Ct^3}$

2.

$$\frac{dy}{dx} = \frac{x^2y^2 + y^2}{x^2y^3 + 4y^2x^2}$$

Key:

$$\frac{dy}{dx} = \frac{x^2y^2 + y^2}{x^2y^3 + 4y^2x^2}$$

$$\frac{dy}{dx} = \frac{(x^2+1)y^2}{x^2(y^3 + 4y^2)}$$

Divide by y^2, dx, and multiply by $y^2 + 4y^2$.

$$\frac{y^3 + 4y^2}{y^2}dy = \frac{x^2+1}{x^2}dx$$

The equation is separable and separated.

Let's integrate both sides:

$$\int \frac{y^3 + 4y^2}{y^2}dy = \int \frac{x^2+1}{x^2}dx$$

$$\int (y+4)dy = \int 1 + \frac{1}{x^2}dx$$

Implicit solution:

$$\frac{y^2}{2} + 4y = x - \frac{1}{x} + C$$

This is a quadratic equation in y, let's solve for y:

$$y^2 + 8y + 16 = x - \frac{1}{x} + \underbrace{(C + 16)}_{C}$$

$$(y+4)^2 = x - \frac{1}{x} + C$$

General solution: $\boxed{y = -4 \pm \sqrt{x - \frac{1}{x} + C}}$

Exercise 3. (6 points) A tank with capacity of 500gal originally contains 200 gallons of water with 100 lb of salt in solution. Water containing 2 lb of salt per gallon is entering at a rate of 5 gal/min, and the mixture is allowed to flow out of the tank at a rate of 3 gal/min.

1. Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow.

   Key: Let $A(t)$ be the amount of salt in the tank, in lb, at time t, in minutes.

$$\frac{dA}{dt} = rate\_in - rate\_out, \qquad rate = flow\_rate * concentration$$

- $rate\_in$: 5gal/min get in with the concentration 2lb/gal.

$$rate\_in = 5 * 2 = 10.$$

- $rate\_out$:3 gal/min get out of the tank. The concentration of salt of the solution that flows out, is the concentration of salt in the tank. The concentration is the quotient $\frac{A(t)}{Volume}$.
  The tank initially contains 200 gallons. Every minute, 5 gal flow in the tank and 3 gallons flow out of the tanks. Every minute, the volume increases by 2 gallons. Therefore,

$$Volume = 200 + 2t.$$
$$rate\_out = 3 * \frac{A(t)}{200 + 2t}$$

therefore A is solution to the initial value problem

$$\frac{dA}{dt} = 10 - \frac{3A(t)}{200 + 2t}, \qquad A(0) = 100$$

Let's solve the initial value problem: The equation is linear and not separable. We'll use the integrating factor method.

$$A' + \frac{3}{200 + 2t} A(t) = 10$$

- Integrating factor:

$$\mu(t) = e^{\int \frac{3}{200+2t}} = e^{\frac{3}{2}\ln(200+2t)} = (200 + 2t)^{3/2}$$

- Multiply by the integrating factor:

$$\underbrace{(200 + 2t)^{3/2} y' + \frac{3A}{(200 + 2t)^{5/2}}}_{\frac{d}{dt}\left((200+2t)^{3/2} A\right)} = 10(200 + 2t)^{3/2}$$

- Integrate both sides:

$$(200 + 2t)^{3/2} A = 10 * \frac{1}{5}(200 + 2t)^{5/2} + C$$

- Divide by the integrating factor:

$$A = 2(200 + 2t) + \frac{C}{(200 + 2t)^{3/2}}$$

- Solve for the initial condition:

$$A(0) = 100 = 400 + \frac{C}{200^{3/2}}$$

$$C = -300(200)^{3/2}$$

- Solution $\boxed{A(t) = 400 + 4t - \frac{300(200)^{3/2}}{(200 + 2t)^{3/2}}}$

2. Find the concentration (in pounds per gallon) of salt in the tank when it is on the point of overflowing.

   Key: Let's find the time when the tank starts to overflow:

   The tank capacity is 500 gallons. The volume of solution is $200 + 2t$. The tank starts to overflow when $200 + 2t = 500$, i.e. after 150 minutes.

   The concentration at time $t = 150$ is

$$\boxed{Concentration = \frac{A(150)}{Volume} = \frac{1000 - \frac{300(200)^{3/2}}{500^{3/2}}}{500} = 2 - \frac{6\sqrt{10}}{125} \approx 1.85 lb/gal}$$

Exercise 4. (5 points)

1. Solve the initial value problem

$$ty' + 2y = 2\frac{e^{3(t-4)}}{t}, \qquad y(4) = 0$$

2. Find $y'(4)$.

Key: The equation is linear, non-separable.

$$y' + \frac{2}{t}y = \frac{2e^{3(t-4)}}{t^2}$$

- Integrating factor:

$$\mu(t) = e^{\int \frac{2}{t}dt} = e^{2\ln(t)} = t^2$$

- Multiply both sides by the integrating factor:

$$\underbrace{t^2y' + 2ty}_{\frac{d}{dt}(t^2y)} = 2e^{3(t-4)}$$

- Integrate both sides:

$$t^2y = \frac{2}{3}e^{3(t-4)} + C$$

- Divide both sides by the integrating factor

$$y = \frac{2e^{3(t-4)}}{3t^2} + \frac{C}{t^2}$$

- Solve for the initial condition:

$$y(4) = 0 = \frac{2}{3 * 16} + \frac{C}{16}$$

$$C = \frac{-2}{3}$$

$$\boxed{y(t) = \frac{2e^{3(t-4)}}{3t^2} - \frac{2}{3t^2}}$$

Use the differential equation to evaluate $y'(4)$:

$$4y'(4) + 2y(4) = \frac{2e^0}{4}$$

Since $y(4) = 0$, $\boxed{y'(4) = \frac{1}{8}}$

An alternative method would be to differentiate the solution found in part 1.

Exercise 5. (4 points) Consider the initial value problem

$$y' = 3 + t - y, \qquad y(1) = 2$$

1. Use Euler method with step size 1/3 to get an approximation of y(0).

   Key: The initial condition is at $t = 1$, and we want an approximation of y at $t = 0$, h will be negative, $h = -1/3$.

n	$t_n = t_{n-1} + h$	$y_n = y_{n-1} + hf(t_{n-1}, y_{n-1})$	$f(t_n, y_n) = 3 + t_n - y_n$
$n = 0$	$t_0 = 1$	$y_0 = 2$	$f(t_0, y_0) = 3 + 1 - 2 = 2$
$n = 1$	$t_1 = 1 - \frac{1}{3} = \frac{2}{3}$	$y_1 = 2 - \frac{1}{3} * 2 = \frac{4}{3}$	$f(t_1, y_1) = 3 + \frac{2}{3} - \frac{4}{3} = \frac{7}{3}$
$n = 2$	$t_2 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$	$y_2 = \frac{4}{3} - \frac{1}{3} * \frac{7}{3} = \frac{5}{9}$	$f(t_2, y_2) = 3 + \frac{1}{3} - \frac{5}{9} = \frac{25}{9}$
$n = 3$	$t_3 = \frac{1}{3} - \frac{1}{3} = 0$	$y_3 = \frac{5}{9} - \frac{1}{3} * \frac{25}{9} = \frac{-10}{27}$	

   An approximation of $y(0)$ is $\boxed{\frac{-10}{27}}$.

2. On the interval [0,1], are the solutions concave up or concave down? Justify your answer.

   Key: Several methods to solve this question:

   - Method 1: $f(t_n y_n)$ is an approximation of the slope of the solution at t_n.
     We found

     $$\begin{aligned} y'(0) &\approx 3 + 0 + \frac{10}{27} \approx 3.4 \\ y'(1/3) &\approx 25/9 \approx 2.8 \\ y'(2/3) &\approx 7/3 \approx 2.3 \\ y'(1) &= 2 \end{aligned}$$

     Clearly the derivative y' is decreasing, therefore the derivative of y' (second derivative) is negative and $\boxed{y \text{ is concave down.}}$

   - Method 2: Find the second derivative of y using the differential equation:
     Take the derivative with respect to t of the differential equation $y' = 3 + t - y$:

     $$y'' = 0 + 1 - y' = 1 - (3 + t - y) = -2 - t + y$$

     y is less than 2 on the interval $[0, 1]$, $t \in [0, 1]$, therefore $y'' < 0$ and y is concave down.