

The introduction of complex numbers in the 16th century was a natural step in a sequence of extensions of the positive integers, starting with the introduction of negative numbers (to solve equations of the form $x + a = 0$), the introduction of rational numbers (to solve equations like $qx + p = 0$, p, q integers) and the introduction of irrational numbers (to solve equations like $x^2 - 2 = 0$). The introduction of $i = \sqrt{-1}$ made it possible to solve the equation $x^2 + 1 = 0$, and in fact any quadratic equation. Pleasantly enough, one does not need any further extensions to solve an arbitrary polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$; such an equation always has n roots (possibly complex and possibly repeated). These notes¹ will present one way of defining complex numbers and familiarize you with some of their properties.

1 The Complex Plane

A *complex number* z is given by a pair of real numbers x and y and is written in the form $z = x + iy$, where i satisfies $i^2 = -1$. The complex numbers may be represented as points in the plane, with the real number 1 represented by the point $(1, 0)$, and the complex number i represented by the point $(0, 1)$. The x -axis is called the “real axis,” and the y -axis is called the “imaginary axis.” For example, the complex numbers 1, i , $3 + 4i$ and $3 - 4i$ are illustrated in FIG 1A.

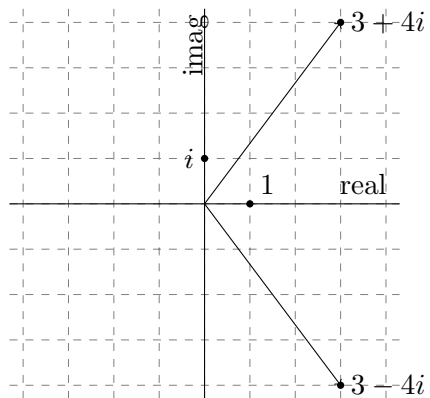


FIG 1A

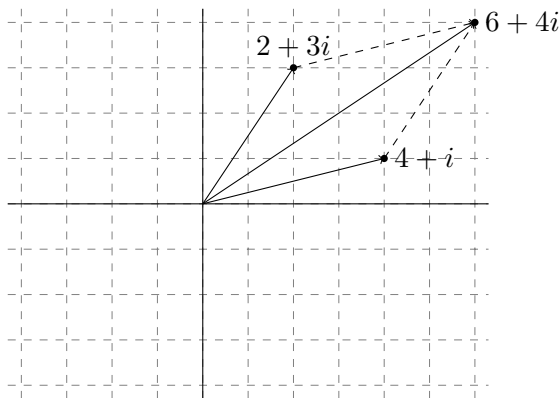


FIG 1B

Complex numbers are added in a natural way: If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \quad (1)$$

It's just vector addition. FIG 1B illustrates the addition $(4 + i) + (2 + 3i) = (6 + 4i)$. Multiplication is given by

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

Note that the product behaves exactly like the product of any two algebraic expressions, keeping in mind that $i^2 = -1$. Thus,

$$(2 + i)(-2 + 4i) = 2(-2) + 8i - 2i + 4i^2 = -8 + 6i$$

¹Based on notes written by Bob Phelps, with modifications by Tom Duchamp and John Palmieri

If $z = x + iy$, then we call x the *real part* of z and y the *imaginary part*, and we write $x = \operatorname{Re} z$, $y = \operatorname{Im} z$. (**Remember:** $\operatorname{Im} z$ is a *real* number.) The term “imaginary” is a historical holdover; it took mathematicians some time to accept the fact that i (for “imaginary,” naturally) was a perfectly good mathematical object. Electrical engineers (who make heavy use of complex numbers) reserve the letter i to denote electric current and they use j for $\sqrt{-1}$.

There is only one way we can have $z_1 = z_2$, namely, if $x_1 = x_2$ and $y_1 = y_2$. An equivalent statement is that $z = 0$ if and only if $\operatorname{Re} z = 0$ and $\operatorname{Im} z = 0$. If a is a real number and $z = x + iy$ is complex, then $az = ax + iay$ (which is exactly what we would get from the multiplication rule above if z_2 were of the form $z_2 = a + i0$). Division is more complicated (although we will show later that the polar representation of complex numbers makes it easy). To find z_1/z_2 it suffices to find $1/z_2$ and then multiply by z_1 . The rule for finding the reciprocal of $z = x + iy$ is given by:

$$\frac{1}{x + iy} = \frac{1}{x + iy} \cdot \frac{x - iy}{x - iy} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{x - iy}{x^2 + y^2} \quad (2)$$

The expression $x - iy$ appears so often and is so useful that it is given a name: it is called the *complex conjugate* of $z = x + iy$, and a shorthand notation for it is \bar{z} ; that is, if $z = x + iy$, then $\bar{z} = x - iy$. For example, $\overline{3 + 4i} = 3 - 4i$, as illustrated in FIG 1A. Note that $\overline{\bar{z}} = z$ and $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$. Exercise (3b) is to show that $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$.

Another important quantity associated with a complex number z is its *modulus* (also known as its *absolute value* or *magnitude*):

$$|z| = (z\bar{z})^{1/2} = \sqrt{x^2 + y^2} = ((\operatorname{Re} z)^2 + (\operatorname{Im} z)^2)^{1/2}$$

Note that $|z|$ is a *real* number. For example, $|3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$. This leads to the inequality

$$\operatorname{Re} z \leq |\operatorname{Re} z| = \sqrt{(\operatorname{Re} z)^2} \leq \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} = |z| \quad (3)$$

Similarly, $\operatorname{Im} z \leq |\operatorname{Im} z| \leq |z|$. Another inequality concerning the modulus is the important *triangle inequality*:

$$|z_1 + z_2| \leq |z_1| + |z_2| \quad (4)$$

To prove this, it suffices to show that the square of the left side is less than the square of the right, so we look at

$$|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = z_1\bar{z}_1 + 2\operatorname{Re}z_1\bar{z}_2 + z_2\bar{z}_2.$$

(The last equality uses Exercise 3 applied to $z_1\bar{z}_2$.) Using the fact (from (3)) that $2\operatorname{Re}z_1\bar{z}_2 \leq 2|z_1\bar{z}_2| = 2|z_1||z_2|$, we get

$$|z_1 + z_2|^2 \leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2 = (|z_1| + |z_2|)^2,$$

which is what we wanted. A useful consequence of the triangle inequality is the following:

$$||z_1| - |z_2|| \leq |z_1 - z_2| \quad (5)$$

Exercises 1.

1. Prove that the product of $z = x + iy$ and the expression (2) above equals 1.

2. Verify each of the following:

$$(a) \quad (\sqrt{2} - i) - i(1 - \sqrt{2}i) = -2i \qquad (b) \quad \frac{1 + 2i}{3 - 4i} + \frac{2 - i}{5i} = -\frac{2}{5}$$

$$(c) \quad \frac{5}{(1 - i)(2 - i)(3 - i)} = \frac{1}{2}i \qquad (d) \quad (1 - i)^4 = -4$$

3. Prove the following:

(a) $z + \bar{z} = 2\operatorname{Re} z$, and z is a real number if and only if $\bar{z} = z$. (Note also that $z - \bar{z} = 2i\operatorname{Im} z$.)

(b) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$.

4. (a) Prove that $|z_1 z_2| = |z_1| |z_2|$ (Hint: Use (3b).)

(b) Prove the inequality in (5). (Hint: by (4), $|z_1| = |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|$.)

5. Find all complex numbers $z = x + iy$ such that $z^2 = 1 + i$.

$$\text{Answer: } z = \pm \left[\sqrt{\frac{\sqrt{2}}{2} + \frac{1}{2}} + i \sqrt{\frac{\sqrt{2}}{2} - \frac{1}{2}} \right].$$

2 Polar Representation of Complex Numbers

Recall that the plane has polar coordinates as well as rectangular coordinates. The relation between the rectangular coordinates (x, y) and the polar coordinates (r, θ) is

$$\begin{aligned} x &= r \cos \theta & \text{and} & & y &= r \sin \theta, \\ r &= \sqrt{x^2 + y^2} & \text{and} & & \theta &= \arctan \frac{y}{x}. \end{aligned}$$

(If $(x, y) = (0, 0)$, then $r = 0$ and θ can be anything.) This means that for the complex number $z = x + iy$, we can write

$$z = r(\cos \theta + i \sin \theta).$$

There is another way to rewrite this expression for z , and it is probably the most important equation in mathematics²:

$$\boxed{\text{Euler's formula: } e^{i\theta} = \cos \theta + i \sin \theta}.$$

²You can derive this if you are familiar with Taylor series. We know that for any real number x , e^x can be expressed as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

For example,

$$e^{i\pi/2} = i, \quad e^{\pi i} = -1 \quad \text{and} \quad e^{2\pi i} = 1,$$

and if you want to combine the five fundamental constants in mathematics into one equation,

$$e^{i\pi} + 1 = 0.$$

Given $z = x + iy$, then z can be written in the form $z = re^{i\theta}$, where

$$r = \sqrt{x^2 + y^2} = |z| \quad \text{and} \quad \theta = \tan^{-1} \frac{y}{x}. \quad (6)$$

That is, r is the magnitude of z . We call $re^{i\theta}$ the *polar representation* of the complex number z .

Note: In the polar representation of complex number, we *always* assume that r is non-negative. The angle θ is sometimes called the *argument* or *phase* of z .

For example, the complex number $z = 8 + 6i$ may also be written as $10e^{i\theta}$, where $\theta = \arctan(.75) \approx 0.64$ radians, as illustrated in FIG 2.

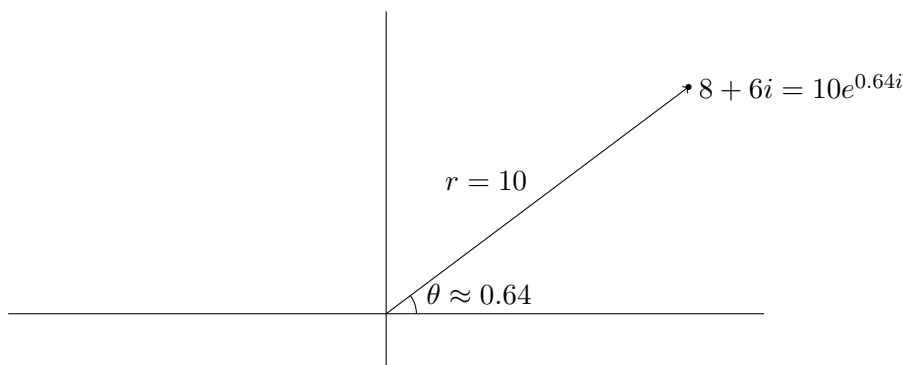


FIG 2

If $z = -4 + 4i$, then $r = \sqrt{4^2 + 4^2} = 4\sqrt{2}$ and $\theta = 3\pi/4$; therefore $z = 4\sqrt{2}e^{3\pi i/4}$. Any angle which differs from $3\pi/4$ by an integer multiple of 2π will give us the same complex number. Thus $-4 + 4i$ can also be written as $4\sqrt{2}e^{11\pi i/4}$ or as $4\sqrt{2}e^{-5\pi i/4}$. In general, if $z = re^{i\theta}$, then we also

For any complex number z , we *define* e^z by the power series

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots .$$

In particular,

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots + \frac{(i\theta)^n}{n!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos \theta + i \sin \theta. \end{aligned}$$

have $z = re^{i(\theta+2\pi k)}$, $k = 0, \pm 1, \pm 2, \dots$. Moreover, there is ambiguity in equation (6) about the inverse tangent which can (and *must*) be resolved by looking at the signs of x and y , respectively, in order to determine the quadrant in which θ lies. If $x = 0$, then the formula for θ makes no sense, but $x = 0$ simply means that z lies on the imaginary axis and so θ must be $\pi/2$ or $3\pi/2$ (depending on whether y is positive or negative).

The conditions for equality of two complex numbers using polar coordinates are not quite as simple as they were for rectangular coordinates. If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then $z_1 = z_2$ if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2\pi k$, $k = 0, \pm 1, \pm 2, \dots$. Despite this, the polar representation is very useful when it comes to multiplication:

$$\text{if } z_1 = r_1e^{i\theta_1} \quad \text{and} \quad z_2 = r_2e^{i\theta_2}, \quad \text{then} \quad z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)} \quad (7)$$

That is, to obtain the product of two complex numbers, *multiply their moduli and add their angles*. To see why this is true, write $z_1z_2 = re^{i\theta}$, so that $r = |z_1z_2| = |z_1||z_2| = r_1r_2$ (by Exercise (4a)). It remains to show that $\theta = \theta_1 + \theta_2$, that is, that $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$ (this is Exercise (7a) below). For example, let

$$z_1 = 2 + i = \sqrt{5}e^{i\theta_1}, \quad \theta_1 \approx 0.464, \quad z_2 = -2 + 4i = \sqrt{20}e^{i\theta_2}, \quad \theta_2 \approx 2.034.$$

If $z_3 = z_1z_2$, then $r_3 = r_1r_2$ and $\theta_3 = \theta_1 + \theta_2$; that is,

$$z_3 = -8 + 6i = \sqrt{100}e^{i\theta_3}, \quad \theta_3 \approx 2.498,$$

as shown in the picture.

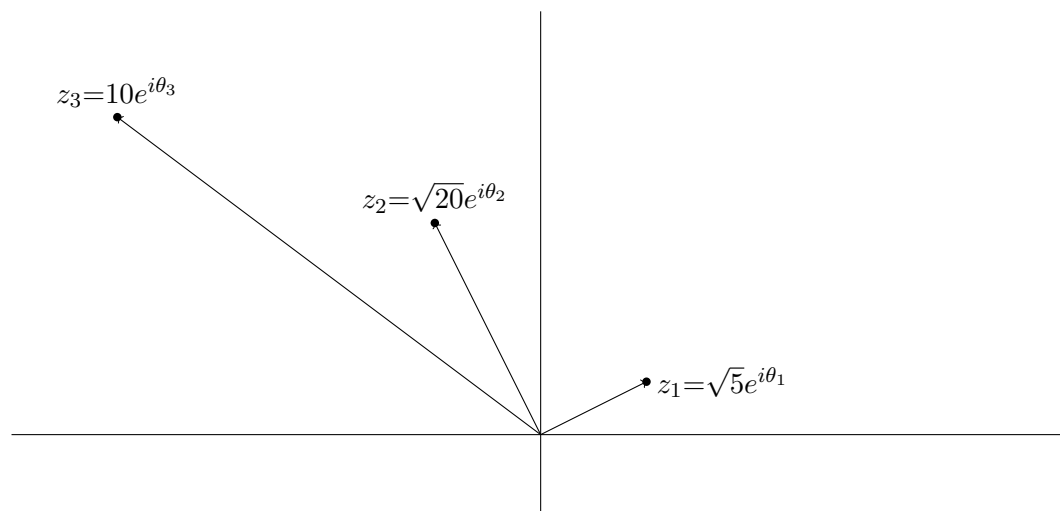


FIG 3

Applying (7) to $z_1 = z_2 = -4 + 4i = 4\sqrt{2}e^{\frac{3}{4}\pi i}$ (our earlier example), we get

$$(4 + 4i)^2 = (4\sqrt{2}e^{\frac{3}{4}\pi i})^2 = 32e^{\frac{3}{2}\pi i} = -32i.$$

The formula in (7) can be used to prove that for any positive integer n ,

$$\text{If } z = re^{i\theta}, \quad \text{then } z^n = r^n e^{in\theta}.$$

This makes it easy to solve equations like $z^4 = -7$. Indeed, writing the unknown number z as $re^{i\theta}$, we have $r^4 e^{i4\theta} = -7 \equiv 7e^{\pi i}$, hence $r^4 = 7$ (so $r = 7^{1/4}$, since r must be a non-negative real number) and $4\theta = \pi + 2k\pi$, $k = 0, \pm 1, \pm 2, \dots$. It follows that $\theta = \pi/4 + 2k\pi/4$, $k = 0, \pm 1, \pm 2, \dots$. There are only four distinct complex numbers of the form $e^{(\pi/4 + k\pi/2)i}$, namely $e^{\pi i/4}$, $e^{3\pi i/4}$, $e^{5\pi i/4}$ and $e^{7\pi i/4}$. Figure 4 illustrates $z = -7$ and its four fourth roots $z_1 = 7^{1/4}e^{\pi i/4}$, $7^{1/4}e^{3\pi i/4}$, $7^{1/4}e^{5\pi i/4}$ and $7^{1/4}e^{7\pi i/4}$, all of which lie on the circle of radius $7^{1/4}$ about the origin.

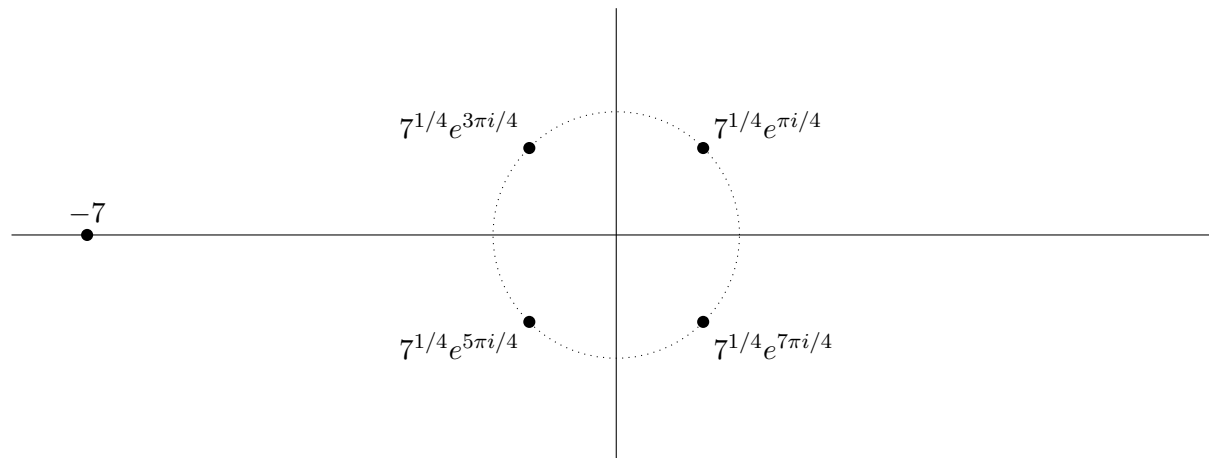


FIG 4

From the fact that $(e^{i\theta})^n = e^{in\theta}$ we obtain De Moivre's formula:

$$(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$$

By expanding on the left and equating real and imaginary parts, you obtain trigonometric identities which can be used to express $\cos n\theta$ and $\sin n\theta$ as a sum of terms of the form $(\cos\theta)^j(\sin\theta)^k$. For example, taking $n = 2$ and looking at the real part produces the familiar formula $\cos 2\theta = \cos^2\theta - \sin^2\theta$. For $n = 3$ one gets $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

Let's also note the following formulas: if $z = re^{i\theta}$, then

$$\bar{z} = re^{-i\theta}, \quad \operatorname{Re} z = r \cos\theta, \quad \operatorname{Im} z = r \sin\theta.$$

Combined with the formulas from Exercise (3a), we get

$$\begin{aligned} \cos\theta &= \frac{1}{2} (e^{i\theta} + e^{-i\theta}), \\ \sin\theta &= \frac{1}{2i} (e^{i\theta} - e^{-i\theta}) = -\frac{i}{2} (e^{i\theta} - e^{-i\theta}). \end{aligned}$$

Exercises 2.

(6) Let $z_1 = 3i$ and $z_2 = 2 - 2i$

(a) Plot the points $z_1 + z_2$, $z_1 - z_2$ and \bar{z}_2 .

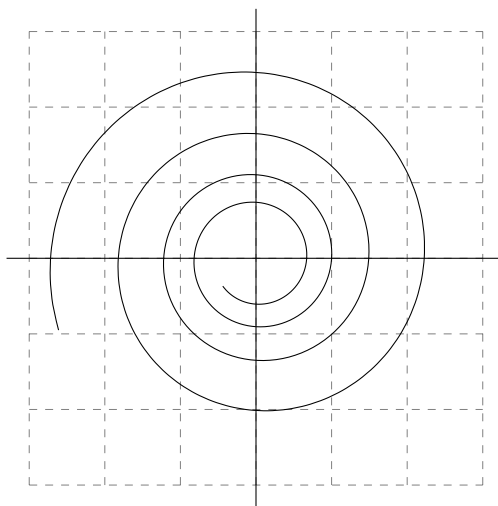
- (b) Compute $|z_1 + z_2|$ and $|z_1 - z_2|$.
- (c) Express z_1 and z_2 in polar form.
- (7) Prove the following:
- (a) $e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$.
 (Hint: this is the same as showing that $(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) = \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)$.)
- (b) Use (a) to show that $(e^{i\theta})^{-1} = e^{-i\theta}$, that is, $e^{-i\theta} e^{i\theta} = 1$.
- (8) Let $z_1 = 6e^{i\pi/3}$ and $z_2 = 2e^{-i\pi/6}$. Plot z_1 , z_2 , $z_1 z_2$ and z_1/z_2 .
- (9) Find all complex numbers z which satisfy $z^3 = -1$.
 (Answer: there are three such numbers: $e^{\pi i/3}$, $e^{i\pi} = -1$, and $e^{5\pi i/3}$.)
- (10) Find all complex numbers z such that $z^2 = \sqrt{2}e^{i\pi/4}$.
 (Answer: $z = 2^{1/4}e^{\pi i/8}$, $2^{1/4}e^{9\pi i/8}$. This is the polar form of the Exercise 5.)

3 The complex exponential function

The *complex exponential function* is “just” an exponential function e^{rt} where r is a complex number. So it looks like

$$e^{(\rho+i\omega)t} = e^{\rho t} e^{i\omega t} = e^{\rho t} \cos(\omega t) + i e^{\rho t} \sin(\omega t).$$

If we let t vary among the real numbers, going from $-\infty$ to ∞ , then the values of this function trace a spiral in the complex plane: the quantity ω is the angular velocity of the spiral ($\omega > 0$ corresponds to a counterclockwise spiral, $\omega < 0$ to a clockwise one). The quantity ρ measures the rate at which the spiral expands outward ($\rho > 0$) or inward ($\rho < 0$).



4 The function $x(t) = e^{\rho t}(C_1 \cos(\omega t) + C_2 \sin(\omega t))$

We want to write the function

$$x(t) = C_1 e^{\rho t} \cos(\omega t) + C_2 e^{\rho t} \sin(\omega t)$$

in the form

$$x(t) = A e^{\rho t} \cos(\omega t - \phi),$$

because then we know what the graph of $x(t)$ looks like.

First notice that

$$A e^{\rho t} \cos(\omega t - \phi) = (A \cos(\phi) \cos(\omega t) + A \sin(\phi) \sin(\omega t)) e^{\rho t},$$

so let

$$A \cos(\phi) = C_1 \text{ and } A \sin(\phi) = C_2.$$

Then we get

$$A = \sqrt{C_1^2 + C_2^2} \text{ and } \tan(\phi) = \frac{C_2}{C_1}.$$

Example 1. Consider the function

$$x(t) = (5 \cos(2t) + 4 \sin(2t)) e^{-t/5}.$$

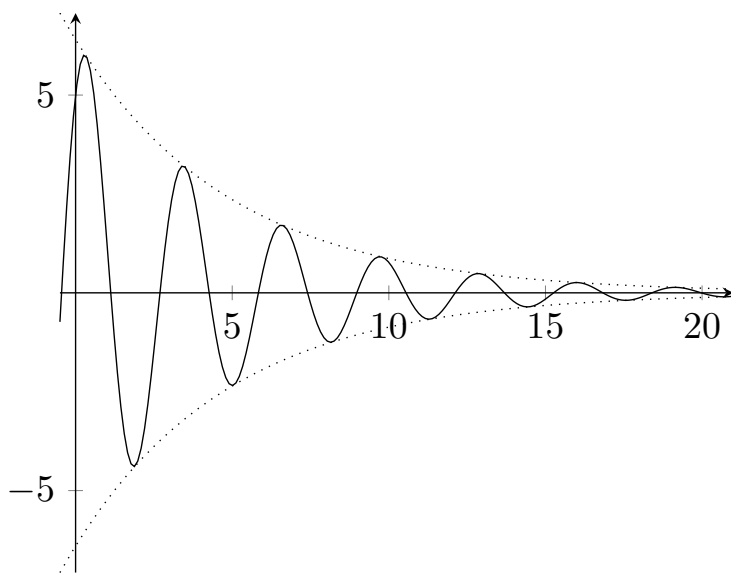
The point $(C_1, C_2) = (5, 4)$ is in the first quadrant so $0 < \phi < \pi/2$. So

$$A = \sqrt{5^2 + 4^2} = \sqrt{41} \text{ and } \phi = \tan^{-1}(4/5).$$

Hence,

$$(5 \cos(2t) + 4 \sin(2t)) e^{-t/5} = \sqrt{41} e^{-t/5} \cos(2t - \tan^{-1}(4/5)).$$

Here is a sketch of this curve, showing it oscillating between $\sqrt{41} e^{-t/5}$ and $-\sqrt{41} e^{-t/5}$:



Note: There is an alternate description of $x(t)$ that makes direct use of the polar form of complex numbers. Since $5 \cos(2t)e^{-t/5}$ is the real part of $5e^{(-1/5+2i)t}$ and since $4 \sin(2t)e^{-t/5}$ is the real part of $-4ie^{(-1/5+2i)t}$, let $C = 5 - 4i$ and $\rho + i\omega = -1/5 + 2i$. Then

$$x(t) = \operatorname{Re} \left((5 - 4i)e^{(-1/5+2i)t} \right)$$

Of course the earlier expression, $\sqrt{41} e^{-t/5} \cos(2t - \tan^{-1}(4/5))$, is easier to graph.

Exercises 3.

(11) Sketch the graph of the curve

$$z(t) = (2 + 2i)e^{(\frac{1}{2} + \pi i)t}$$

for $0 \leq t \leq 3$. Sketch the graph of $x = x(t) = \operatorname{Re}(z(t))$.

(12) Consider the function

$$x(t) = 3e^{-2t} \cos(4t) - 5e^{-2t} \sin(4t).$$

Write it in each of the forms

$$x(t) = Ae^{\rho t} \cos(\omega t - \phi)$$

and

$$x(t) = \operatorname{Re} (Ce^{rt})$$

where A , ω and ϕ are real numbers and C and r are complex numbers.