Worksheet 3 Math 126

(best if done in small groups)

Precalculus courses do not always include material ellipses and hyperbolas. This worksheet covers material that will be useful in section 12.6, for example.

The curve given by the the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

is called an **ellipse**.

1. Why is $|x| \leq |a|$ and $|y| \leq |b|$ for (x, y) on the ellipse (1)? This means that the graph of the ellipse is inside a rectangle. Where does the ellipse touch the edges of the rectangle?

2. Fix a number $L > 2$. Then all points (x, y) in the plane such that the distance from $(-1, 0)$ to (x, y) plus the distance from (x, y) to $(1, 0)$ equals L is an ellipse. Use the distance formula to prove this statement. The points $(-1, 0)$ and $(1, 0)$ are called **foci** of the ellipse. Changing the foci, of course, changes the ellipse. Hint: square the sum of the distances, move everything except the remaining square root to one side of the equation, then square again.

3. The mathematician Richard Arens (the mathematical great-grandfather of one of our faculty members, and former professor at UCLA) wanted a contractor to build his swimming pool in the shape of an ellipse. The contractor became rather frustrated trying to copy the drawing onto the ground, so Arens attached a string to two stakes in the ground, pulling the string tight using a third stake. He dragged the third stake along the ground while keeping the string tight (as the string slid around the stake). Try this for yourself: use a piece of string attached to two thumb-tacks (or have a friend hold the ends on a piece of paper), and use a pencil instead of the third stake.

4. It is too time consuming to use the technique of Exercise 3 in practice. A quick way to draw an ellipse is to use the information derived in Exercise 1, and roughly sketch an oval with the correct intersections with the coordinate axes. Try this with the ellipse given by
$$4x^2 - y^2/4 = 1.$$

A curve given by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{2}$$

is called a **hyperbola**. Actually there are two curves which satisfy equation (2), and they are called the "branches" of the hyperbola.

5. (a) Why is $|x| \geq |a|$ on this hyperbola? What does this statement mean about the graph of the hyperbola?
   (b) Show that the branches of the hyperbola lie between the two lines $y = ax/b$ and $y = -ax/b$ and are asymptotic to both lines.
   Hint: You may assume both a and b are positive. Multiply equation (2) by b^2/x^2. If $x \geq a$, then show that $y = ax/b$ is too big and $y = -ax/b$ is too small to satisfy (2). Then let $x^2 \to \infty$.

A geometric description of a hyperbola: Fix a number $L > 1$. Then all points (x, y) in the plane such that the distance from $(-1, 0)$ to (x, y) minus the distance from (x, y) to $(1, 0)$ equals L is a branch of a hyperbola. This fact can be proved using the distance formula much like exercise 2.

6. A curve given by the equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \tag{3}$$

   is also a hyperbola. How is the graph of (3) related to the graph of (2)? You can tell which graph is which by noting that y cannot equal zero in (3) and x cannot equal zero in (2).

7. Graph the hyperbola $4x^2 - 9y^2 = 1$. Also draw the asymptotic lines, and label the closest points to $(0, 0)$ (these are called "vertices").

One of the greatest intellectual achievements of the human race was to show that the path of a planet, or "wandering star", can be described as an ellipse with the sun at one of the foci. A space vehicle passing by a planet has a path which is (approximately) a hyperbola. The planet actually can act like a slit shot for the space vehicle, increasing the speed of the space vehicle as it passes by.

There is one more "conic section", which you have already seen, called a **parabola**. For example the curve given by the equation $y = ax^2$ is a parabola.

If you sped through this worksheet because you already knew the material and still have time left in quiz section, try the following exercise (it is not needed for subsequent material in the course, but it is interesting and it utilizes some skills you learned in math 124):

8. Draw a parabola, $y = ax^2$ with $a > 0$. Then draw a vertical line from high above the x-axis until it hits the parabola at a point P. This line intersects the parabola in an (acute) angle θ. Draw another line segment from P to a point Q on the y axis so that this line segment also intersects the parabola at P with the same (acute) angle. If you rolled a billard ball along the vertical line on paper toward the parabola, it will bounce off a wall represented by the parabola and then follow the second line segment. Find the point Q on the y-axis. Hint: write $P = (x_0, ax_0^2)$. Then $\cot\theta$ is the slope of the tangent line at P. Relate the slope of the segment PQ to θ, and use it to find the equation of the line containing PQ.

Notice that Q in exercise 8 does *not* depend on the choice of P (it does depend on the parabola given by ax^2). The point Q is called the **focus** of the parabola. Light coming from a source far away, will roughly travel in parallel lines. If you aim the parabola at the light source, every light "ray" hitting the parabola will bounce off the parabola and hit the focus. If you construct a surface which is the rotation of a parabola about its central axis, then it can be used to focus sunlight, indeed start a fire. Satellite dishes are constructed this way, putting the pickup for the receiver at the focus. Flashlights are also constructed this way, putting the bulb at the focus, sending the light out along a narrow beam.

Parabolas, hyperbolas and ellipses are sometimes called conic sections. Figure 1 in Section 10.5 of Stewart shows how to obtain conic sections by cutting a double cone with a plane.