Worksheet on Electric Fields Math 126

The electric field due to a charge q measured at a distance d from the charge is given by

$$E(d) = \frac{q}{d^2}.$$

Electric fields are additive, so that charges $q_1, q_2, \ldots, q_n$ located at points $x_1, x_2, \ldots, x_n$ on the real line give an electric field

$$E(x) = \sum_{k=0}^{n} \frac{q_k}{(x - x_k)^2}, \tag{1}$$

measured at x. If a continuous charge density $\rho(x)$ is spread over the interval $[-1, 1]$, then the electric field measured at x is given by

$$E(x) = \int_{-1}^{1} \frac{\rho(t)}{(x-t)^2} dt. \tag{2}$$

(a) Write out a Riemann sum for (2) and observe that it is a sum of the form (1), with charges $\rho(x_j)\Delta x$ located at x_j.

(b) Using Taylor's Estimate, show that

$$\frac{1}{(1-u)^2} = \sum_{k=0}^{\infty} (k+1)u^k,$$

for small u.

(c) Fix t, $-1 \leq t \leq 1$, and use part (b) to show that for large x

$$\frac{1}{(x-t)^2} = \frac{1}{x^2} \cdot \frac{1}{(1 - \frac{t}{x})^2} = \frac{1}{x^2} \sum_{k=0}^{\infty} (k+1) \left(\frac{t}{x}\right)^k.$$

(d) Use your estimate from part (b) to show that

$$E(x) = \lim_{n \to \infty} \sum_{k=2}^{n} \frac{\int_{-1}^{1} (k-1) t^{k-2} \rho(t) dt}{x^k}.$$

(e) Writing out the first few terms for $E(x)$:

$$E(x) = \frac{\int_{-1}^{1} \rho(t) dt}{x^2} + \frac{2 \int_{-1}^{1} t\rho(t) dt}{x^3} + \frac{3 \int_{-1}^{1} t^2 \rho(t) dt}{x^4} + \ldots.$$

The integrals in the sum above are called the *moments* of the charge density. Approximate the coefficient of x^{-2} using a Riemann sum, and use it to explain why that coefficient is called the total charge.

(f) Charges can be negative, so it is possible that the total charge is zero. Show that the electric field is approximately the total charge divided by x^2 for large x, if the total charge is not zero. If the total charge is zero and the first moment (coefficient of x^{-3}) is non-zero, show that the electric field is approximately proportional to $\frac{1}{x^3}$ for large x (this is similar to problem 5 in HW#2.)