Worksheet 1, Math 126
Applications of Linear Approximation.

Introduction. The main purpose of this worksheet is to show how the Linear (Tangent Line) Approximation may be used to estimate the solution of an equation in a semi-realistic example. In particular, we will develop a way to use linear approximation repeatedly to improve the estimate, and use the Tangent Line Error Bound to bound the error of the estimate. Secondary purposes include review of differentiation techniques and introduction of the idea of a numerical method. This worksheet is best done in small groups.

NPR has a radio show called CarTalk. They received the following question, which they could not answer: A truck driver's gas gauge was broken and he wanted to just look into his tank to see when he had 1/4 tank left so that he would know it's time to think about getting gas. The usual gas tank on a semi-truck is a cylinder on its side, and you can tell how full it is by putting a stick into it. From Math 125, you know that the volume of gas in the tank is equal to the cross sectional area times the length of the cylinder. So you need to know when the cross sectional area is one-quarter of the area of the circular cross section of the tank.

1. Draw a picture of circle of radius r that just touches the x axis but otherwise is above it. Shade the region G which is inside the circle and below the line $y = h$, the height of the gas in the tank. We will use the fact that the shaded area is

$$\text{Area}(G) = r^2\Big[\frac{\pi}{2} - (1 - \frac{h}{r})\sqrt{\frac{2h}{r} - (\frac{h}{r})^2} - \sin^{-1}(1 - \frac{h}{r})\Big].$$

You should be able to verify this using techniques you learned in math 125, but for now, just assume it is true. Let $c = h/r$ be the ratio of the height of the gas to the radius of the circle. Show that if the area of G is one quarter of the area of the circle then c is the solution to the equation

$$\frac{\pi}{4} - (1 - x)\sqrt{2x - x^2} - \sin^{-1}(1 - x) = 0.$$

2. If $f(x)$ is the left–hand side of the above equation, we'd like to find x so that $f(x) = 0$. Check that $f(x)$ is positive when $x = 1$ (when the height of the gas equals the radius) and that it is negative when $x = 1/2$ (the height of the gas is half of the radius). The solution should be between $\frac{1}{2}$ and 1.

3. The function f is too complicated to actually find the solution x, so we will use the first Taylor polynomial. Find the first Taylor polynomial $T_1(x)$ for f based at $b = \frac{1}{2}$, and solve the equation $T_1(x) = 0$. (The original function f is approximately equal to T_1, so f will be approximately 0 at this value of x. We chose $b = \frac{1}{2}$ because it was the best guess from problem 2.)

Newton's Method. We've just seen an example of a problem in which we needed to find a value r (called a "zero" or "root") such that $f(r) = 0$. For simple functions like linear or quadratic polynomials, there is a formula for the zeros. For more complicated functions $f(x)$, usually no such formula is possible. We can replace f by its linear approximation, and find where the linear approximation equals zero as we did above. But for many practical problems, we need a more accurate approximation. The rest of this worksheet introduces a numerical method, Newton's Method, for finding a zero of a function. Like most numerical methods, it gives an approximate answer whose accuracy can be improved by repeating similar calculations.

4. Let f be a differentiable function. (You can think of the function f above as an example, but do the following calculations for the general case.) We are going to find the first Taylor polynomial for f based at a sequence of different points x_1, x_2, etc. Suppose x_1 is given, and let $T_1(x)$ be the first Taylor polynomial for f based at $b = x_1$. We will choose x_2 to be the zero of T_1 (and thus approximately a zero of f). Show that if $T_1(x_2) = 0$, then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Discussion: Newton's idea was to repeat this process. If x_2 is approximately a zero of f, find the first Taylor polynomial based at $b = x_2$. Then by the same reasoning as you just did to find x_2, the new T_1 will equal zero at

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$

Repeating the idea of step 3 for $n = 1, 2, 3, \ldots$, set

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Then the numbers $x_1, x_2, x_3, \ldots$ should get closer and closer to a zero r of f, i.e. $f(r) = 0$.

5. Let's illustrate this with a simpler function, $f(x) = x^2 - 3$, starting with $x_1 = 2$. First draw the picture. Graph f and T_1 based at $b = 2$, and label the zeros of f and the zero of T_1. The zero of T_1 is the x-value of the point where the tangent line to the graph of f crosses the x axis. Calculate this value.

   We can interpret your picture in the following way: We made a guess for the zero r of f, in this case we guessed 2. Then the tangent line to the graph of f based at $b = 2$ crosses the x axis at a point much closer to the zero of f than 2, namely at $x_2 = 1.75$. In other words, we've improved our guess. Repeat this process starting with 1.75 instead of 2: Graph the tangent line to the curve $y = x^2 - 3$ based at $b = 1.75$. (You may need to draw a new, larger scale graph.) Find where this line crosses the x-axis. This number, x_3, should be even closer to $\sqrt{3} = 1.732\ldots$.

6. We can use some algebra and a calculator to find the sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

If $f(x) = x^2 - 3$, we can simplify the right side to

$$x_{n+1} = \frac{x_n}{2} + \frac{3}{2x_n}.$$

Set $x_1 = 2$, and use your calculator to find x_2, x_3, x_4 and x_5. Notice that these numbers are getting close to $\sqrt{3}$, which is a zero of f.

7. Now we return to the general case to discuss the accuracy of x_1, x_2, x_3, ..., x_n as approximations for a zero of f. Suppose that we are given a function f and an initial guess b for a zero of f. Let $c = b - f(b)/f'(b)$. Suppose $f(r) = 0$ and suppose that I is an interval containing r, b and c. Suppose also that $|f''(x)| \leq M$ and $|f'(x)| \geq K$ for all x in the interval I. Use the Tangent Line Error Bound for f based at b (setting $x = r$) to show that

$$|c - r| \leq \frac{M}{2K}|b - r|^2.$$

We can apply this error bound at each stage of Newton's method. The error bound means that if the error in the guess (x_n) for the zero (r) at stage n is at most 10^{-m}, then the error at stage $n + 1$ is at most $(M/2K)10^{-2m}$. In other words, the number of correct digits roughly doubles at each step. This is remarkably fast convergence! How many digits are correct in x_1, x_2, x_3, x_4, and x_5 in problem 5 above?

Some concluding remarks. Applying Newton's method to the gas tank problem is a bit more complicated than the calculations you have just done, and you are probably running out of time in this quiz section. From the simpler example of $f(x) = x^2 - 3$, you can see how Newton's Method gives a good approximation of the zero of a function, and how the error bound decreases dramatically after each successive step. More important applications include, for example, solving Kepler's equation which arises in the study of planetary motion. Some calculators have a "Solve" button which is used to find a zero, that is, solve an equation. These calculators use Newton's method, except that the derivative is estimated by difference quotients instead of computed analytically. If you have time at home, test your understanding by applying a few steps of Newton's method to the gas tank problem. Check your work by evaluating f at each step. The values $f(x_n)$ should become very small, and the number of digits of x_n which are the same as the digits of the root r should roughly double at each step.

A listener to CarTalk suggested a quick approximate solution to the gas tank problem. Drink a can of pop until there is 1/4 left. You can measure the depth with a straw when the can is standing up. Then turn the can on its side and measure the depth of the pop and find the ratio with the radius (half of the diameter) of the can. This gives a rough approximation for the ratio the truck driver should use (about 3:5).