1. **[12 points]** Find all critical points of the function $f(x, y) = x^4 - 18x^2 - y^2 + 6y$.

Classify each critical point as a local minimum, local maximum, or saddle point.

$f_x(x,y) = 4x^3 - 36x = 0 \rightarrow 4x(x^2-9) = 0 \rightarrow x = 0, 3, \text{or } -3$

$f_y(x,y) = -2y + 6 = 0 \rightarrow y = 3$

$f_{xx}(x,y) = 12x^2 - 36$

$f_{yy}(x,y) = -2$

$f_{xy}(x,y) = 0$

negative

$D(0,3) = (-36)(-2) - 0^2 = 72$

$D(-3,3) = (72)(-2) - 0^2 = -144$

$D(3,3) = (72)(-2) - 0^2 = -144$

local max @ $(0,3)$

saddle pts @ $(3,3)$ & $(-3,3)$

2. **[4 points]** Give an example of a function that has only one saddle point, located at $(1, 2, 3)$.

You do not need to fully justify your answer. Coming up with an example is enough.

This problem is worth 4 points. If you're stuck, try the other problems and come back to it later!

One possible answer:

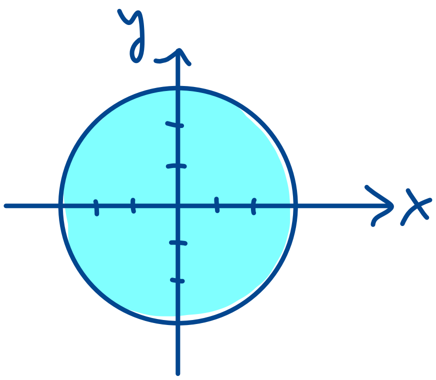
$f(x,y) = x^2 - y^2$ has a saddle pt at $(0,0,0)$, so shift it to get

$$\boxed{f(x,y) = (x-1)^2 - (y-2)^2 + 3}$$

(Lots of other answers also work)

3. **[14 points]** Let $\mathcal{D}$ be the disc of radius 3 centered at the origin.

Find the absolute minimum and maximum values of $f(x, y) = x^2 - 2y^2 - x^2y$ on $\mathcal{D}$.



Critical pts:

$f_x(x,y) = 2x - 2xy = 0 \rightarrow 2x(1-y) = 0$

$x = 0$ or $y = 1$

$f_y(x,y) = -4y - x^2 = 0$

$-4y = 0$, $y = 0$

$x^2 = -4$, no solutions

So $(0,0)$ is the only critical point

Points to check:

$f(0,0) = 0$

$f(0,-3) = -18$

$f(0,3) = -18$

$f(\sqrt{8}, -1) = 14$

$f(-\sqrt{8}, -1) = 14$

Boundary:

$x^2 + y^2 = 9 \rightarrow x^2 = 9 - y^2$

$f(x,y) = (9-y^2) - 2y^2 - (9-y^2)y = y^3 - 3y^2 - 9y + 9$

where $-3 \le y \le 3$

$f'(y) = 3y^2 - 6y - 9 = 0$

$y^2 - 2y - 3 = 0$

$(y-3)(y+1) = 0$

$y = 3$ or $y = -1$

$y = 3 \rightarrow x = 0$

$y = -1 \rightarrow x^2 = 8 \rightarrow x = \pm\sqrt{8}$

$\boxed{\text{Abs min } -18 \quad \text{Abs max } 14}$

4. **[8 points]** Suppose the plane tangent to the surface $z = f(x, y)$ at $(3, 2, 3)$ passes through the points $(2, 4, 1)$ and $(1, 2, -3)$.

Find $f_x(3, 2)$ and $f_y(3, 2)$.

One way

Find a plane through these points

$(3,2,3)$ to $(2,4,1)$: $\langle -1, 2, -2 \rangle$

$(3,2,3)$ to $(1,2,-3)$ $\langle -2, 0, -6 \rangle$

cross product: $\langle -12, -2, 4 \rangle$

$-12x - 2y + 4z = -28$

$$z = \underset{f_x}{3}x + \underset{f_y}{\tfrac{1}{2}}y - 7$$

$$\boxed{f_x(3,2) = 3 \quad f_y(3,2) = \tfrac{1}{2}}$$

Or

$z = f_x(3,2)(x-3) + f_y(3,2)(y-2) + 3$

$(2,4,1)$: $1 = -f_x(3,2) + 2f_y(3,2) + 3$

$(1,2,-3)$: $-3 = -2f_x(3,2) + 3$

$$\boxed{f_x(3,2) = 3 \qquad f_y(3,2) = \tfrac{1}{2}}$$

5. **[8 points]** Compute the integral $\int_0^2 \int_0^3 y(\sin(xy) + 3)\, dy\, dx$.

Fubini's thm: $\rightarrow \int_0^3 \int_0^2 y(\sin(xy) + 3)\, dx\, dy$

$u = xy \quad du = y\,dx$

$$= \int_0^3 \int_0^{2y} (\sin(u) + 3)\, du\, dy = \int_0^3 \Big(-\cos(u) + 3u\Big)\Big]_{u=0}^{u=2y} dy$$

$$= \int_0^3 \left(-\cos(2y) + 6y + 1\right) dy = \left(-\tfrac{1}{2}\sin(2y) + 3y^2 + y\right)\Big]_0^3$$

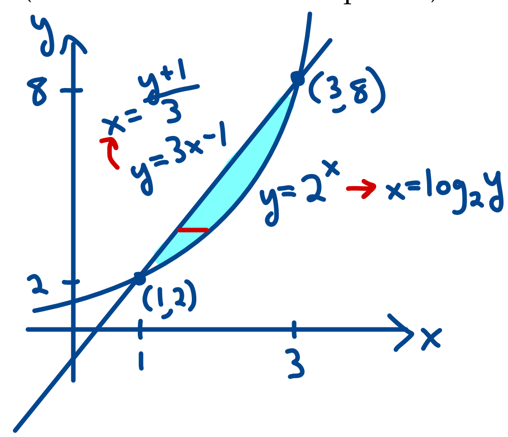
$$= \boxed{-\tfrac{1}{2}\sin(6) + 30}$$

6. **[7 points per part]** Rewrite the following integrals as instructed.

Do not try to evaluate these integrals! Just rewrite them.

(a) $\displaystyle\int_1^3 \int_{2^x}^{3x-1} \sqrt{5x+7}\cos(xy)\,dy\,dx.$ Reverse the order of integration.

(That's 2 raised to the x^{th} power.)



$$\boxed{\int_2^8 \int_{\frac{y+1}{3}}^{\log_2 y} \sqrt{5x+7}\cos(xy)\,dx\,dy}$$

(b) $\displaystyle\int_0^3 \int_x^{\sqrt{6x-x^2}} \sin(\sqrt{x^2+y^2})\,dy\,dx.$ Convert this to a polar integral.

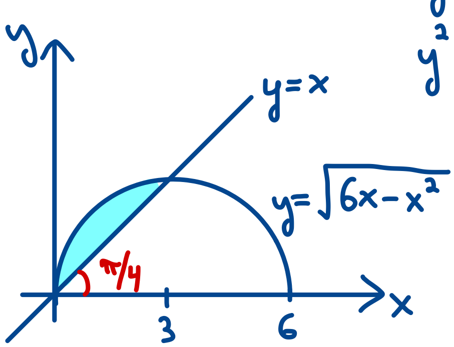
$y = \sqrt{6x-x^2}$ circle of radius 3 centered @ (3,0)

$y^2 = 6x - x^2 \longrightarrow x^2+y^2 = 6x$

$r^2 = 6r\cos\theta$

$r = 6\cos\theta$

$\sin(\sqrt{x^2+y^2}) \to \sin(r)$



$$\boxed{\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{6\cos\theta} \sin(r)\,r\,dr\,d\theta}$$