Math 126, Sections C and D, Winter 2014, Midterm II

February 25, 2014

Name Solutions

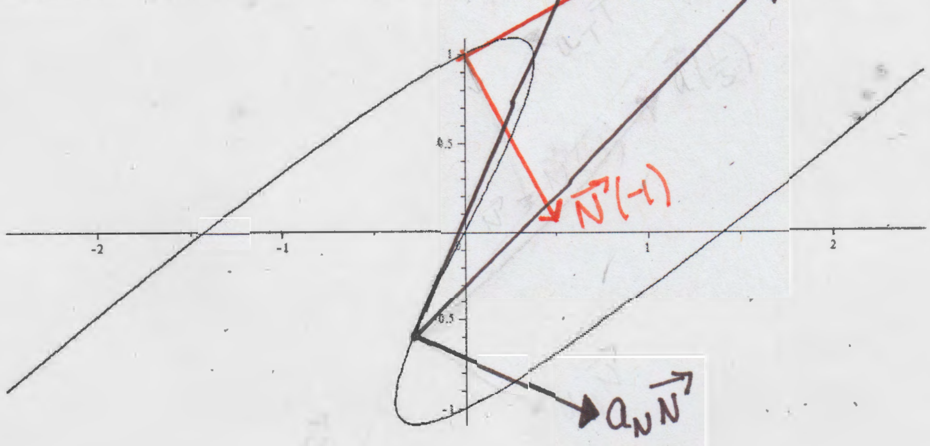
TA/Section

Instructions.

- There are 4 questions. The exam is out of 40 points.
- You are allowed to use one page of notes written only on one side of the sheet in your own handwriting. **Hand in your notes with your exam paper.**
- You may use a calculator which does not graph and which is not programmable. Even if you have a calculator, give me exact answers. ($\frac{2\ln 3}{\pi}$ is exact, 0.7 is an approximation for the same number.)
- **Show your work.** If I cannot read or follow your work, I cannot grade it. You may not get full credit for a right answer if your answer is not justified by your work. If you continue at the back of a page, make a note for me. Please $\boxed{\text{BOX}}$ your final answer.

Question	points
1	
2	
3	
4	
Total	

1. The vector function $\mathbf{r}(t) = \langle t^3 - t, t^3 - 2t, 0\rangle$ sketches the following curve on the xy-plane:



(a) Compute the unit tangent vector T at the point where $t = -1$. Sketch $\mathbf{T}(-1)$ and $\mathbf{N}(-1)$ at that point on the graph above.

$\vec{r}\,'(t) = \langle 3t^2-1, 3t^2-2, 0\rangle$ $\quad \vec{r}\,'(-1) = \langle 2, 1, 0\rangle$ $\quad |\vec{r}\,'(-1)| = \sqrt{4+1} = \sqrt{5}$

$\vec{T}(-1) = \langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\rangle$ $\quad \vec{r}(-1) = \langle -1+1, -1+2, 0\rangle = \langle 0, 1, 0\rangle$

(b) Compute the acceleration vector $\mathbf{a}(t)$ and its normal and tangential components at $t = \frac{1}{3}$. Sketch the acceleration vector $\mathbf{a}\left(\frac{1}{3}\right))$ together with $a_N\mathbf{N}$ and $a_T\mathbf{T}$ at the point where $t = \frac{1}{3}$ on the graph above.

$\vec{a}(t) = \vec{r}\,''(t) = \langle 6t, 6t, 0\rangle$ $\quad \vec{a}\left(\frac{1}{3}\right) = \langle 2, 2, 0\rangle$

$\vec{r}\left(\frac{1}{3}\right) = \langle \frac{1}{27} - \frac{1}{3}, \frac{1}{27} - \frac{2}{3}, 0\rangle = \langle \frac{-8}{27}, \frac{-17}{27}, 0\rangle \approx \langle -0.3, -0.6, 0\rangle$

$\vec{r}\,'\left(\frac{1}{3}\right) = \langle \frac{3}{9} - 1, \frac{3}{9} - 2, 0\rangle = \langle -\frac{6}{9}, -\frac{15}{9}, 0\rangle = \langle -\frac{2}{3}, -\frac{5}{3}\rangle$

$\vec{r}\,''\left(\frac{1}{3}\right) = \langle 2, 2, 0\rangle$ $\quad \vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} i & j & k \\ -\frac{2}{3} & -\frac{5}{3} & 0 \\ 2 & 2 & 0 \end{vmatrix} = \left(-\frac{4}{3} + \frac{10}{3}\right)\vec{k} = \langle 0, 0, 2\rangle$

$\vec{r}\,' \cdot \vec{r}\,'' = \langle -\frac{2}{3}, -\frac{5}{3}, 0\rangle \cdot \langle 2, 2, 0\rangle = -\frac{14}{3}$

$a_N = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|} = \frac{2}{\sqrt{\frac{4}{9} + \frac{25}{9}}} = \frac{6}{\sqrt{29}} \approx 1.1$ $\quad a_T = \frac{\vec{r}\,' \cdot \vec{r}\,''}{|\vec{r}\,'|} = \frac{-14/3}{\frac{\sqrt{29}}{9}} = \frac{-14}{\sqrt{29}} \approx -2.6$

(c) Without any further computation, find the Binormal vectors at these two points:

$\mathbf{B}(-1) = -\vec{k}$ $\qquad \mathbf{B}\left(\frac{1}{3}\right) = \vec{k}$

2. Answer the following about $xy^3 + yz^3 + zx^3 = 3$.

(a) Compute the partial derivatives z_x and z_y. Remember that this is impicit differentiation and you have to treat z like a function.

z_x: $y^3 + y \cdot 3z^2 z_x + z_x x^3 + z \cdot 3x^2 = 0$

$$z_x = \frac{-y^3 - 3x^2 z}{3z^2 + x^3}$$

z_y: $3xy^2 + z^3 + y \cdot 3z^2 z_y + z_y x^3 = 0$

$$z_y = \frac{-3xy^2 - z^3}{3yz^2 + x^3}$$

(b) Find the equation of the tangent plane to the surface given by $xy^3 + yz^3 + zx^3 = 3$ at the point $(1, 1, 1)$.

at $x=1, y=1, z=1$ $\quad z_x = \frac{-4}{4} = -1 \quad z_y = \frac{-4}{4} = -1$

$$z - 1 = -(x-1) - (y-1)$$
$$z = -x - y + 3$$

(c) Use linear approximation to find the value of z when $x = 0.93$ and $y = 1.06$.

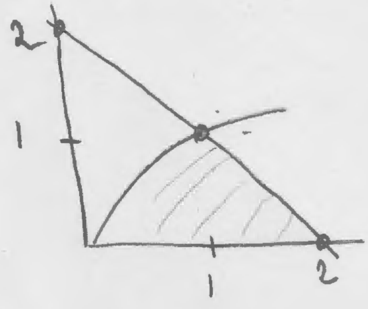
$$z \simeq -0.93 - 1.06 + 3 = -1.99 + 3 = 1.01$$

3. Given the double integral

$$\int\int_D xy^2 \; dA$$

where D is the region below $y = \sqrt{x}$, above the x axis and to the left of the line $y = -x + 2$,

(a) Sketch the region labeling all intersection points.



$\sqrt{x} = -x+2$

$x = x^2 - 4x + 4$

$0 = x^2 - 5x + 4$

$0 = (x-4)(x-1)$

(b) Set up the integral in $dxdy$ order. You may have to split the region.

$$\int_0^1 \int_{y^2}^{-y+2} xy^2 \; dx\, dy$$

(c) Set up the integral in $dydx$ order. You may have to split the region.

$$\int_0^1 \int_0^{\sqrt{x}} xy^2 \, dy\, dx + \int_1^2 \int_0^{-x+2} xy^2 \, dy \, dx$$

(d) Evaluate one of your answers in (b) or (c) (or both if you have extra time).

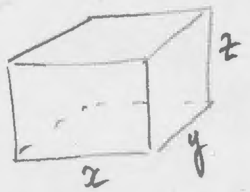
$$(b) = \int_0^1 \left. \frac{x^2y^2}{2} \right|_{y^2}^{-y+2} dy = \int_0^1 \frac{y^2}{2}\left[(-y+2)^2 - y^4\right] dy$$

$$= \int_0^1 \frac{y^2}{2}\left[y^2 - 4y + 4 - y^4\right] dy = \int \frac{y^4}{2} - 2y^3 + 2y^2 - \frac{y^6}{2} \, dy$$

$$= \left. \frac{y^5}{10} - \frac{2y^4}{4} + \frac{2y^3}{3} - \frac{y^7}{14} \right|_0^1 = \frac{1}{10} - \frac{1}{2} + \frac{2}{3} - \frac{1}{14} = \frac{21 - 105 + 140 - 15}{2 \cdot 5 \cdot 3 \cdot 7}$$

$$= \frac{41}{210}$$

4. The base of an aquarium with volume 800 cubic meters is made of slate and the sides are made of glass. If slate costs 36 dollars per square meter and the glass costs 12 dollars per square meter, find the dimensions of the aquarium that minimize the cost of the materials.



$800 = xyz \longrightarrow z = 800/xy$

$\text{Cost} = 36xy + 12(2yz + 2xz)$

$$C(x,y) = 36xy + 24\left(y \cdot \frac{800}{xy} + x \cdot \frac{800}{xy}\right)$$

$$= 36xy + 24\left(\frac{800}{x} + \frac{800}{y}\right)$$

$$C_x = 36y - \frac{24 \cdot 800}{x^2} = 0 \longrightarrow x^2 y = \frac{\overset{8\,2}{\cancel{24}} \cdot 800}{\underset{12\,3}{\cancel{36}}} = \frac{1600}{3}$$

$$C_y = 36x - \frac{24 \cdot 800}{y^2} = 0 \longrightarrow xy^2 = \frac{160}{3}$$

so $x^2 y = y^2 x$ so $x = y$

$x^2 y = x^3 = \frac{1600}{3}$ so $x = y = \left(\frac{1600}{3}\right)^{1/3} \approx$

and $z = \frac{800}{\left(\frac{1600}{3}\right)^{2/3}}$

So the dimensions are:

Base is a square with side $\left(\frac{1600}{3}\right)^{1/3} \approx 8.1$

Depth is $\frac{800}{\left(\frac{1600}{3}\right)^{2/3}} \approx 12.2$

To make sure it is a minimum:

$C_{xx} = \frac{48(800)}{x^3} = \frac{48(800)}{\left(\frac{1600}{3}\right)} = 72 > 0$ $\quad C_{yy} = \frac{48(800)}{y^3} = 72$

$C_{xy} = 36 \qquad D = 72^2 - 36^2 > 0$