1. Consider the function

$$f(x, y) = xy^2 + \sqrt{y - x} - e^{x-2}.$$

(a) **[6 points]** Write the linearization $L(x, y)$ for f at the point $(2, 3)$.

$$f_x(x,y) = y^2 - \frac{1}{2\sqrt{y-x}} - e^{x-2}$$

$$f_x(2,3) = 9 - \frac{1}{2} - 1 = 7.5$$

$$f_y(x,y) = 2xy + \frac{1}{2\sqrt{y-x}}$$

$$f_y(2,3) = 12 + \frac{1}{2} = 12.5$$

$$f(2,3) = 18 + 1 - 1 = 18$$

$$L(x,y) = 7.5(x-2) + 12.5(y-3) + 18$$

or

$$\boxed{L(x,y) = 7.5x + 12.5y - 34.5}$$

(b) **[3 points]** Use your answer from part (a) to find an *approximate* solution to

$$\underbrace{9.61}_{y^2}x + \sqrt{\underbrace{3.1}_{y} - x} - e^{x-2} = 19.1.$$

$$L(x, 3.1) = 19.1$$

$$7.5x + 12.5(3.1) - 34.5 = 19.1$$

$$\boxed{x = 1.98}$$

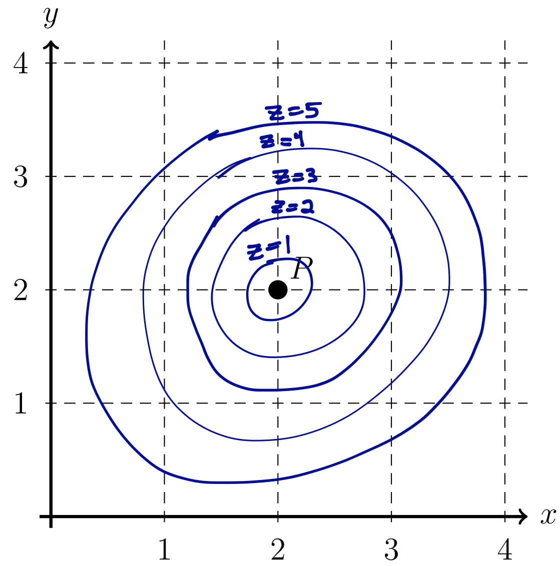
2. **[3 points each]**

Due to budget cuts, you'll have to draw your own graphs for this problem.

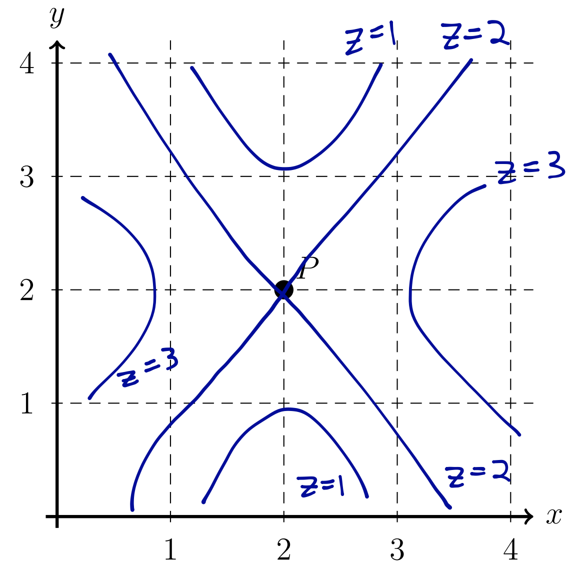
For each part (a)–(d), draw some level curves for a surface $z = f(x, y)$ satisfying the given conditions at the point $P = (2, 2)$. (Assume f is a continuous and differentiable function.)

In order to get full credit, you must draw enough level curves *and label them* so that I can actually see the indicated features in the graph.

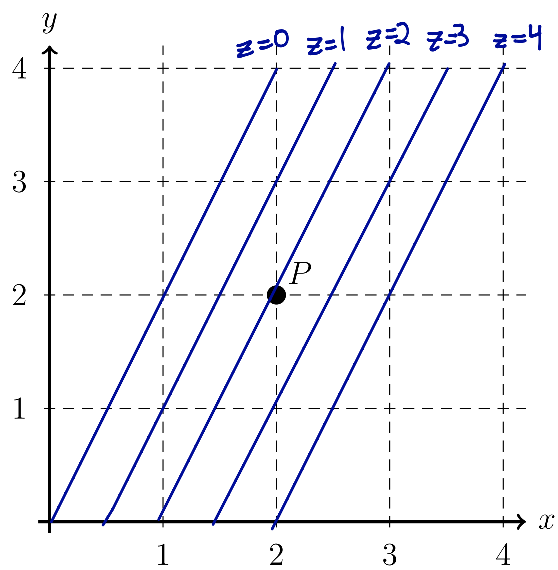
(a) There's a local minimum at P.



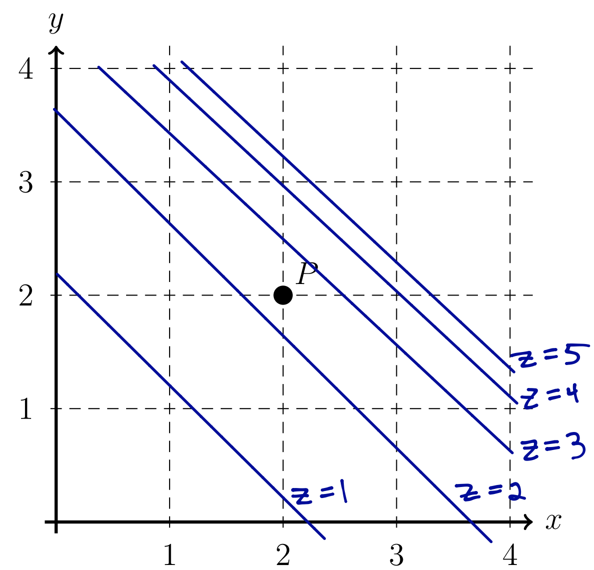
(b) There's a saddle point at P.



(c) $f_x(P) = 2$, and $f_y(P) = -1$.



(d) $f_x(P)$, $f_y(P)$, $f_{xx}(P)$, and $f_{yy}(P)$ are all positive.



(other answers are possible)

3. **[10 points]** Find the points on the hyperbolic paraboloid $\underbrace{x = z^2 - y^2}$ that are closest to the point $(1, 3, 0)$.

For full credit, you must justify your answer!

$z^2 = x + y^2$

Dist. from (x, y, z) to $(1, 3, 0)$:

$$\sqrt{(x-1)^2 + (y-3)^2 + z^2} = \sqrt{(x-1)^2 + (y-3)^2 + x + y^2}$$

$$f(x, y) = (\text{dist})^2 = (x-1)^2 + (y-3)^2 + x + y^2$$

$$f_x(x, y) = 2(x-1) + 1 = 0 \rightarrow x = \frac{1}{2}$$

$$f_y(x, y) = 2(y-3) + 2y = 0 \rightarrow y = \frac{3}{2}$$

Only one crit. point: $\left(\frac{1}{2}, \frac{3}{2}\right)$.

Second deriv. test:

$f_{xx}(x, y) = 2$

$f_{xy}(x, y) = 0$

$f_{yy}(x, y) = 4$

$D\left(\frac{1}{2}, \frac{3}{2}\right) = 8 \rightarrow$ local min!

So, closest point is @ $x = \frac{1}{2}$, $y = \frac{3}{2}$.

$z^2 = x + y^2 = \frac{11}{4}$, so $z = \pm \frac{\sqrt{11}}{2}$

$$\boxed{\left(\frac{1}{2}, \frac{3}{2}, \frac{\sqrt{11}}{2}\right) \ \& \ \left(\frac{1}{2}, \frac{3}{2}, -\frac{\sqrt{11}}{2}\right)}$$

4. **[6 points per part]** Happy Thursday! Please evaluate these integrals.

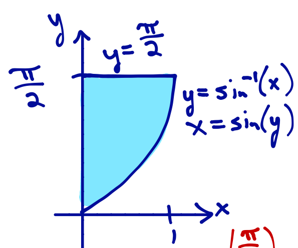
(a) $\displaystyle\int_0^3 \int_4^5 y \sin(xy)\, dy\, dx$

Easier to compute

$$\int_4^5 \int_0^3 y \sin(xy)\, dx\, dy = \int_4^5 \left(-\cos(xy) \right]_{x=0}^{x=3} dy = \int_4^5 \left(-\cos(3y) + 1 \right) dy$$

$$= \left(\frac{-1}{3} \sin(3y) + y \right]_4^5 = \boxed{\frac{-1}{3}\left(\sin(15) - \sin(12) \right) + 1}$$

(b) $\displaystyle\int_0^1 \int_{\sin^{-1}(x)}^{\frac{\pi}{2}} e^{\cos(y)}\, dy\, dx$



$$= \int_0^{\frac{\pi}{2}} \int_0^{\sin(y)} e^{\cos(y)}\, dx\, dy$$

$$= \int_0^{\frac{\pi}{2}} \left(e^{\cos(y)} x \right]_{x=0}^{x=\sin(y)} dy = \int_0^{\frac{\pi}{2}} \sin(y) e^{\cos(y)}\, dy = -\int_1^0 e^u\, du = -\left(e^0 - e^1 \right) = \boxed{e - 1}$$

$\cos\left(\frac{\pi}{2}\right)$

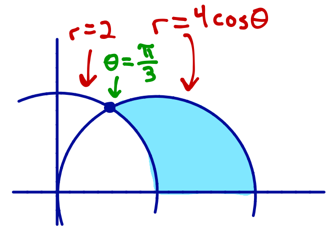
$\cos(0)$

$u = \cos(y)$

$du = -\sin(y)\, dy$

5. **[8 points]** Let R be the region in the first quadrant inside the circle $(x-2)^2+y^2=4$ and outside the circle $x^2+y^2=4$. Compute

$$\iint_R 3y\,dA.$$



Polar!

$$\int_0^{\frac{\pi}{3}}\int_2^{4\cos\theta} 3r^2\sin\theta\,dr\,d\theta$$

$$=\int_0^{\frac{\pi}{3}}\sin\theta\left(r^3\right)\Big]_{r=2}^{r=4\cos\theta}d\theta=\int_0^{\frac{\pi}{3}}\sin\theta\left(64\cos^3\theta-8\right)d\theta$$

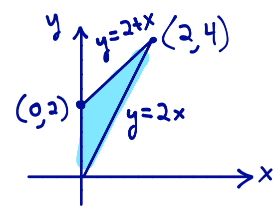
$u=\cos\theta$

$du=-\sin\theta\,d\theta$

$$=-\int_1^{\frac{1}{2}}\left(64u^3-8\right)du=-\left(16u^4-8u\right)\Big]_1^{1/2}=-\left((1-4)-(16-8)\right)=\boxed{11}$$

6. **[9 points]** A lamina is in the shape of a triangle with vertices $(0,0)$, $(0,2)$, and $(2,4)$, with density proportional to the $\underbrace{\text{distance to the } x\text{-axis}}$. $\rho(x,y)=y$

Find the center of mass of the lamina.



$$\text{Mass}=\int_0^2\int_{2x}^{2+x} y\,dy\,dx=\int_0^2\left(\frac{1}{2}y^2\right)\Big]_{2x}^{2+x}dx$$

$$=\int_0^2\left(2+2x-\frac{3}{2}x^2\right)dx=\left(2x+x^2-\frac{1}{2}x^3\right)\Big]_0^2=4$$

$$M_x=\int_0^2\int_{2x}^{2+x} y^2\,dy\,dx=\int_0^2\left(\frac{1}{3}y^3\right)\Big]_{2x}^{2+x}dx=\int_0^2\frac{1}{3}\left(8+12x+6x^2-7x^3\right)dx=\frac{1}{3}\left(8x+6x^2+2x^3-\frac{7}{4}x^4\right)\Big]_0^2=\frac{28}{3}$$

$$M_y=\int_0^2\int_{2x}^{2+x} xy\,dy\,dx=\int_0^2\left(\frac{1}{2}xy^2\right)\Big]_{2x}^{2+x}dx=\int_0^2\left(2x+2x^2-\frac{3}{2}x^3\right)dx=\left(x^2+\frac{2}{3}x^3-\frac{3}{8}x^4\right)\Big]_0^2=\frac{10}{3}$$

So, center of mass is $\left(\dfrac{\frac{10}{3}}{4},\ \dfrac{\frac{28}{3}}{4}\right)=\boxed{\left(\frac{5}{6},\frac{7}{3}\right)}$